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A unified concept of approximate and quasi efficient solutions and associated subdifferentials in multiobjective optimization

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Dedicated to Marco A. López on the occasion of his 70th birthday

Abstract In this paper, we introduce some new notions of quasi efficiency and quasi proper efficiency for multiobjective optimization problems that reduce to the most important concepts of approximate and quasi efficient solutions given up to now. We establish main properties and provide characterizations for these solutions by linear and nonlinear scalarizations. With the help of quasi efficient solutions, a generalized subdifferential of a vector mapping is introduced, which generates a number of approximate subdifferentials frequently used in optimization in a unifying way. The generalized subdifferential is related to the classical subdifferential of real functions by the method of scalarization. An application of generalized subdifferential to express optimality conditions for quasi efficient solutions is also given.

Keywords Multiobjective optimization · Quasi efficiency · Linear scalarization · Nonlinear scalarization · Vector subdifferential

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1 Introduction

The first notion of quasi efficiency was introduced by Loridan [33] for multiobjective optimization problems with the Pareto order. The aim of studying quasi efficient solutions is to obtain feasible points whose objective value is close to be optimal. The introduction of these types of solutions stems from the Ekeland variational principle [8], which determines that an ε-solution is close to an exact solution of a related optimization problem with a perturbed objective mapping. There are several works dedicated to quasi solutions, as for instance [1,7,22,38] for scalar optimization problems, [11,32,33] for multiobjective optimization problems, and [12,19,20,24] for vector optimization. In particular, in [20], the concept of quasi efficiency is extended by considering approximate quasi weak/proper/efficient solutions of vector optimization problems.

In order to establish conditions for approximate efficient solutions, it is necessary to develop new concepts of subdifferentials. Since the introduction by Brøndsted-Rockafellar [4] of the ε-subdifferential for an extended real-valued convex mapping, there have been several extensions in the scalar and the vector cases which provide a powerful tool to derive optimality conditions for different types of approximate solutions in optimization (see, for instance, [6,13,15,31, 36]). A common point in all of these subdifferentials is that the precision error is fixed. On the other hand, it is clear that given a nonconvex scalar mapping, one cannot guarantee the existence of a hyperplane supporting its epigraph at a given point, so it is possible that this function is not subdifferentiable in the classical sense. In [2,3], with the aim of studying stability and duality of nonconvex problems, Azimov and Gasimov introduced and studied a weak subdifferential for scalar mappings, which is a more relaxed concept, in the sense that the supporting hyperplanes are replaced by conic surfaces, defined by means of suitable concave functions. The approximation error is measured, in some sense, by a function involved in the definition of the supporting conic surfaces. In other words, it depends on the decision variable, and so it is not a fixed error. Later, in [30], Küçük et al. extended the scalar weak subdifferential to the vector case and studied its properties and some existence conditions.

In the present paper, following the idea of [2,3] we use the error depending on the decision variable to extend, in the framework of multiobjective optimization, the notion of quasi efficiency given in [20] and the notion of weak subdifferential of [30] to a more general setting. The new definition of quasi efficiency unifies almost all notions of approximate and quasi efficient solutions known in the recent literature. As an application of the new concept of quasi efficient solutions we introduce generalized subdifferentials and characterize them through linear and nonlinear scalarization.

It is worth to mention that all the concepts and results of this paper can be extended to the framework of vector optimization, but we prefer to present them for multiobjective optimization problems as in this setting they are more illustrative without loss of generality.

The paper is structured as follows. In Section 2 we give some preliminaries. In Section 3 we introduce the notions of quasi efficiency and quasi proper efficiency, that extend and unify the notions of quasi efficiency cited above, and we establish important properties of these generalized solutions. Then, in Section 4 we provide characterizations for generalized solutions by linear and nonlinear scalarizations. Section 5 presents an application of the new notion of quasi efficient solutions. Namely, we use these solutions to introduce efficient and proper subddiferentials for vector mappings. We study some of their properties, and compare them with classical subdifferential of real valued functions by scalarization. It is worthwhile mentioning that almost all subdifferentials of vector mappings we use in vector optimization can be obtained from the newly introduced subdifferentials by specifying the error in a suitable form. Finally, we present an existence result for proper efficient subdifferential and optimality conditions for quasi solutions in terms of these generalized subdifferentials. In Section 6 we give some conclusions.

2 Preliminaries

Given a nonempty set $M \subset \mathbb{R}^n$, we denote by $\operatorname{int} M$, $\operatorname{cl} M$, M^c and $\operatorname{cone} M$ the topological interior, the closure, the complement and the cone generated by M, respectively. When $\operatorname{int} M$ is nonempty, we say that M is solid. It is said to be coradiant if $M \supset \bigcup_{\alpha>1} \alpha M$. Coradiant sets provide useful tools in optimization, in particular in duality theory (see [40,44]).

We consider on $\mathbb{R}^n$ a partial order defined by a cone D in the usual way,

$$y_1 \leq_D y_2 \Longleftrightarrow y_2 - y_1 \in D,\ \forall y_1, y_2 \in \mathbb{R}^n.$$

We suppose that D is closed, convex and pointed ($D \cap (-D) = \{0\}$). Also, when D is solid we define

$$y_1 <_D y_2 \Longleftrightarrow y_2 - y_1 \in \operatorname{int} D,\ \forall y_1, y_2 \in \mathbb{R}^n.$$

The set of nonnegative scalars is denoted as $\mathbb{R}_+$ and we denote $\overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$. The positive and the strict positive polar cones of D are denoted, respectively, by D^+ and D^{s+}, that is,

$$D^+ = \{\mu \in \mathbb{R}^n : \langle \mu, d\rangle \geq 0,\ \forall d \in D\},$$
$$D^{s+} = \{\mu \in \mathbb{R}^n : \langle \mu, d\rangle > 0,\ \forall d \in D\backslash\{0\}\}.$$

In the present work we consider the following constrained multiobjective optimization problem

$$\operatorname{Min}\{f(x) : x \in S\}, \qquad (\mathcal{P})$$

where $f = (f_1, f_2, \ldots, f_n) : \mathbb{R}^m \to \overline{\mathbb{R}}^n$ is the objective mapping and $\emptyset \neq S \subset \mathbb{R}^m$ is the feasible set. We suppose that $S \subset \operatorname{dom} f := \{x \in \mathbb{R}^m : f(x) \in \mathbb{R}^n\}$.

For problem $(\mathcal{P})$, we recall standard definitions of efficiency (see, for instance, [23,25,28,42]). A point $x_0 \in S$ is said to be

(i) an efficient solution of $(\mathcal{P})$ if there is no $x \in S$ such that $f(x) \leq_D f(x_0)$, $f(x) \neq f(x_0)$;
(ii) a weak efficient solution of $(\mathcal{P})$ if $\operatorname{int} D \neq \emptyset$ and there is no $x \in S$ such that $f(x) <_D f(x_0)$;
(iii) a proper efficient solution of $(\mathcal{P})$ in the sense of Henig if there exists a proper (i.e., different from $\mathbb{R}^n$), solid and convex cone $D' \subset \mathbb{R}^n$ such that $D\backslash\{0\} \subset \operatorname{int} D'$ and there is no $x \in S$ such that $f(x) <_{D'} f(x_0)$.

The sets of efficient, weak efficient and proper efficient solutions are respectively denoted by $\mathrm{E}(f, S, D)$, $\mathrm{WE}(f, S, D)$ and $\mathrm{He}(f, S, D)$. It is clear that $\mathrm{He}(f, S, D) \subset \mathrm{E}(f, S, D) \subset \mathrm{WE}(f, S, D)$.

Given a scalar mapping $l : \mathbb{R}^m \to \overline{\mathbb{R}}$ and $\varepsilon \geq 0$, we also remind the classical Brøndsted-Rockafellar ε-subdifferential of l at $x_0 \in \operatorname{dom} l$ (see [4]), which is defined by

$$\partial_\varepsilon l(x_0) = \{x^* \in \mathbb{R}^m : l(x) \geq l(x_0) - \varepsilon + \langle x^*, x - x_0\rangle,\ \forall x \in \mathbb{R}^m\}.$$

For $\varepsilon = 0$, it reduces to the classical subdifferential of l at x_0 and is denoted by $\partial l(x_0)$. We recall also the concept of weak subdifferential introduced by Azimov and Gasimov in [2].

Definition 1 The weak subdifferential of l at $x_0 \in \operatorname{dom} l$ is defined as

$$\begin{aligned}\partial^w l(x_0) := \{(x^*, c) \in \mathbb{R}^m \times \mathbb{R}_+ :& \\ & l(x) \geq l(x_0) + \langle x^*, x - x_0\rangle - c\|x - x_0\|,\ \forall x \in \mathbb{R}^m\}.\end{aligned}$$

It follows that $x^* \in \partial l(x_0)$ if and only if $(x^*, 0) \in \partial^w l(x_0)$, and $(x^*, c) \in \partial^w l(x_0)$ if and only if x_0 is a solution of problem $\mathrm{Min}\{l(x) - \langle x^*, x\rangle + c\|x - x_0\| : x \in \mathbb{R}^m\}$.

Also, if l is convex, it follows that $(x^*, c) \in \partial^w l(x_0)$ if and only if $x^* \in \hat{\partial}_c l(x_0)$, where $\hat{\partial}_c l(x_0)$ is the analytic c-subdifferential of l at x_0 introduced by Dinh et al in [6], and used to define the basic (also called limiting or Mordukhovich) subdifferential of l at x_0 via a sequential regularization (see [6] and the references therein).

3 Quasi efficiency

When solving a multiobjective/vector optimization problem, the most familiar kinds of solutions that mathematicians are interested in, are proper, efficient and weak efficient solutions. The choice of a respective type of solution depends on the characteristics of the problem and the necessities of the solver. In this section we give generalizations of the corresponding notions of proper, efficient and weak efficient solutions for problem $(\mathcal{P})$ that unify the most known concepts of exact, approximate and quasi efficiency given in the literature.

From now on, we consider a function $h : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}_+$ such that $h(x, z) \geq 0$, for all $x, z \in \mathbb{R}^m$, $h(x, z) > 0$ whenever $x \neq z$, and introduce the following sets

$$\mathcal{H} := \{\emptyset \neq C \subset \mathbb{R}^n \backslash \{0\} : C \cap (-D) = \emptyset\},$$
$$\overline{\mathcal{H}} := \{\emptyset \neq C \subset \mathbb{R}^n \backslash \{0\} : \operatorname{cl}\operatorname{cone} C \cap (-D \backslash \{0\}) = \emptyset\},$$
$$\mathcal{G}(C) := \left\{ \begin{array}{c} D' \subset \mathbb{R}^n \text{ solid pointed convex cone such that} \\ D \backslash \{0\} \subset \operatorname{int} D' \text{ and } C \cap (-\operatorname{int} D') = \emptyset \end{array} \right\}.$$

Definition 2 Let $x_0 \in S$ and $C \in \mathcal{H}$.
(i) It is said that x_0 is a (C, h)-quasi efficient solution of $(\mathcal{P})$, and we denote it by $x_0 \in \mathrm{QE}(f, S, C, h)$, if there is no $x \in S \backslash \{x_0\}$ such that

$$f(x_0) \in f(x) + h(x, x_0)C.$$

(ii) If C is solid, it is said that $x_0 \in S$ is a (C, h)-quasi weak efficient solution of $(\mathcal{P})$, and we denote it by $x_0 \in \mathrm{QWE}(f, S, C, h)$, if there is no $x \in S \backslash \{x_0\}$ such that

$$f(x_0) \in f(x) + h(x, x_0) \operatorname{int} C.$$

Definition 3 Let $x_0 \in S$ and $C \in \overline{\mathcal{H}}$. It is said that x_0 is a (C, h)-quasi proper efficient solution of $(\mathcal{P})$, and we denote it by $x_0 \in \mathrm{QPE}(f, S, C, h)$, if one can find some $D' \in \mathcal{G}(C)$ such that there is no $x \in S \backslash \{x_0\}$ such that

$$f(x_0) \in f(x) + h(x, x_0)(C + \operatorname{int} D').$$

Equivalent definitions can be given as follows: A point $x_0 \in S$ is (C, h)-quasi efficient (respectively (C, h)-quasi weak efficient/ (C, h)-quasi proper efficient) if and only if

$$f(x) + h(x, x_0)c - f(x_0) \neq 0, \ \forall x \in S \backslash \{x_0\}, \forall c \in C$$

(respectively, $\forall c \in \operatorname{int} C / \ \forall c \in C + \operatorname{int} D'$ with some $D' \in \mathcal{G}(C)$.)
It is also clear from the definitions that when C is solid,

$$x_0 \in \mathrm{QWE}(f, S, C, h) \Leftrightarrow x_0 \in \mathrm{QE}(f, S, \operatorname{int} C, h), \tag{1}$$

and when $C \in \overline{\mathcal{H}}$,

$$x_0 \in \mathrm{QPE}(f, S, C, h) \Leftrightarrow x_0 \in \mathrm{QE}(f, S, C + \operatorname{int} D', h) \text{ for some } D' \in \mathcal{G}(C). \tag{2}$$

Moreover, if $C, C' \in \mathcal{H}$ and $C' \subset C$, then

$$\mathrm{QE}(f, S, C, h) \subset \mathrm{QE}(f, S, C', h). \tag{3}$$

This inclusion generalizes Proposition 2.4 of [34] (Chapter 2) when $C \cup \{0\}$ is a cone and $C' \cup \{0\}$ is a subcone of $C \cup \{0\}$.

Proposition 1 *Let $C \in \mathcal{H}$ be given. The following statements hold.*

(i) If C is solid, then

$$\mathrm{QE}(f,S,C,h) \subset \mathrm{QWE}(f,S,C,h).$$

Equality holds provided C is open.

(ii) If $C \in \overline{\mathcal{H}}$, then

$$\begin{aligned}\mathrm{QPE}(f,S,C,h) &= \bigcup_{D' \in \mathcal{G}(C)} \mathrm{QE}(f,S,C+\operatorname{int} D',h) \\ &\subset \mathrm{QE}(f,S,C+D\backslash\{0\},h).\end{aligned} \tag{4}$$

Moreover, $\mathrm{QPE}(f,S,C,h) = \mathrm{QPE}(f,S,C+D,h)$.

(iii) If $C \in \overline{\mathcal{H}}$ and $D' \in \mathcal{G}(C)$, then

$$\mathrm{QPE}(f,S,C+\operatorname{int} D',h) = \mathrm{QE}(f,S,C+\operatorname{int} D',h).$$

In particular, $\mathrm{QPE}(f,S,C,h) = \bigcup_{D' \in \mathcal{G}(C)} \mathrm{QPE}(f,S,C+\operatorname{int} D',h)$.

Proof For $C \in \mathcal{H}$, one has $\operatorname{int} C \subset C$ and $\operatorname{int} C \in \mathcal{H}$. In view of (3) and (1),

$$\mathrm{QE}(f,S,C,h) \subset \mathrm{QE}(f,S,\operatorname{int} C,h) = \mathrm{QWE}(f,S,C,h),$$

which gives the inclusion in (i). Equality is immediate from (1) if C is open. Furthermore, equality (4) follows directly from (2). The inclusion in (ii) is due to (3) and to the fact that $C + D \setminus \{0\} \subset C + \operatorname{int} D'$ for every $D' \in \mathcal{G}(C)$. To prove (iii), let us apply (4) for $C + \operatorname{int} D'$ instead of C:

$$\mathrm{QPE}(f,S,C+\operatorname{int} D',h) = \bigcup_{D'' \in \mathcal{G}(C+\operatorname{int} D')} \mathrm{QE}(f,S,C+\operatorname{int} D'+\operatorname{int} D'',h). \tag{5}$$

Because $D' \in \mathcal{G}(C+\operatorname{int} D')$ and $C+\operatorname{int} D' \subset (C+\operatorname{int} D'+\operatorname{int} D'')$ for all $D'' \in \mathcal{G}(C+\operatorname{int} D')$ we obtain from (3) that the set $\mathrm{QE}(f,S,C+\operatorname{int} D',h)$ is a member of the union on the right hand side of (5) and contains all members of that union. Consequently,

$$\bigcup_{D'' \in \mathcal{G}(C+\operatorname{int} D')} \mathrm{QE}(f,S,C+\operatorname{int} D'+\operatorname{int} D'',h) = \mathrm{QE}(f,S,C+\operatorname{int} D',h)$$

establishing the first equality in (iii). This and (4) yield the second part of (iii).

We recall (see, for instance [5,18]) that a set C is an improvement set (with respect to the ordering cone D) if $C \subset \mathbb{R}^n\backslash\{0\}$ and $C + D = C$. Equality $\mathrm{QPE}(f,S,C,h) = \mathrm{QPE}(f,S,C+D,h)$ tells us that computing (C,h)-quasi proper solutions can be done by using only improvement sets. Improvement sets are often considered as approximations of the ordering cone. Colloquially speaking, an improvement set is a cone with a hole around the point 0, with which one can obtain suitable sets of approximate solutions when exact solutions are difficult to find. Improvement sets and coradiant sets are closely related to each other. It is known that convex and coradiant sets are improvement sets with respect to their positive hulls (see, for instance, [17, Lemma 3.1]). A partial converse statement is sometimes true too. Namely, if an improvement set C is contained in $D\backslash\{0\}$, then it is coradiant because for every $\alpha > 1$, one has $\alpha C \subset C + (\alpha-1)C \subset C + D = C$.

Remark 1 (a) When $C = D\backslash\{0\}$, we have $h(x,x_0)C = D\backslash\{0\}$, $h(x,x_0)\operatorname{int} C = \operatorname{int} D$ and $h(x,x_0)(C + \operatorname{int} D') = \operatorname{int} D'$ for all $x \in S\backslash\{x_0\}$, and hence, Definitions 2, 3 reduce to the usual notions of efficient, weakly efficient, and proper efficient solutions.
(b) When $h(x,z) = \varepsilon > 0$, for all $x, z \in \mathbb{R}^m$, the notion of (C,h)-quasi efficiency (resp., weak and proper efficiency), is nothing but the concept of (C,ε)-efficiency (resp., weak efficiency, proper efficiency in the sense of Henig), introduced in [17, Definition 3.2] (resp., [17], [15, Definition 3.1]).
The set of (C,ε)-efficient (resp., weak and proper efficient) solutions is denoted by $\mathrm{AE}(f,S,C,\varepsilon)$ (resp., $\mathrm{WAE}(f,S,C,\varepsilon)$, $\mathrm{He}(f,S,C,\varepsilon)$).

We recall also the concept of generalized quasi efficiency introduced in [20, Definition 3.1].

Definition 4 Let $\varepsilon \geq 0$, $G \in \mathcal{H}$ such that G is convex and $0 \notin \operatorname{cl} G$, and let $\varphi : \mathbb{R}^m \to \mathbb{R}_+$ be such that $\varphi(x) > 0$ for $x \neq 0$. A point $x_0 \in S$ is said to be a generalized ε-quasi minimizer of problem $(\mathcal{P})$ with respect to the pair (G,φ) if

$$f(x) + \varepsilon\varphi(x - x_0)e - f(x_0) \notin -D\backslash\{0\}, \ \forall x \in S, \ \forall e \in G \tag{6}$$

or equivalently, there is no $x \in S\backslash\{x_0\}$ such that

$$f(x_0) \in f(x) + \varepsilon\varphi(x - x_0)G + D\backslash\{0\}.$$

If D is solid and (6) holds for $\operatorname{int} D$ instead of $D\backslash\{0\}$, then it is said that x_0 is a weak generalized ε-quasi minimizer of $(\mathcal{P})$. Also, if there exists $K \in \mathcal{G}(G)$ such that (6) is verified for K instead of D, then x_0 is said to be a proper generalized ε-quasi minimizer of $(\mathcal{P})$.

In the following proposition, we relate Definitions 2, 3 and 4.

Proposition 2 *Let $x_0 \in S$, $\varepsilon > 0$, $G \in \mathcal{H}$ such that G is convex and $0 \notin \operatorname{cl} G$, and let $\varphi : \mathbb{R}^m \to \mathbb{R}_+$ be such that $\varphi(x) > 0$ for $x \neq 0$. Let $h(x,z) = \varepsilon\varphi(x-z)$. The following statements hold.*

(a) If $C = G + D\backslash\{0\}$ (resp., $C = G + \operatorname{int} D$), then the notion of (C,h)-quasi efficient (resp., quasi weak efficient) solution reduces to the concept of generalized (resp., weak generalized) ε-quasi minimizer with respect to the pair (G,φ).
(b) If $C = G \in \overline{\mathcal{H}}$, then the notion of (C,h)-quasi proper efficient solution is equivalent to the concept of proper generalized ε-quasi minimizer with respect to the pair (G,φ).

Proof First, it is easy to check that $C \in \mathcal{H}$ using the fact that $G \in \mathcal{H}$. Observe that since D is a convex cone, for every strictly positive number t, one has $t(G + D\backslash\{0\}) = tG + D\backslash\{0\}$ and $t(G + \operatorname{int} D) = tG + \operatorname{int} D$. Therefore, for $x \in S\backslash\{x_0\}$, relation $f(x_0) \in f(x) + \varepsilon\varphi(x - x_0)G + D\backslash\{0\}$ (resp., $f(x_0) \in f(x) + \varepsilon\varphi(x - x_0)G + \operatorname{int} D$) holds if and only if $f(x_0) \in f(x) + h(x, x_0)C$ (resp., $f(x_0) \in f(x) + h(x, x_0)(G + \operatorname{int} D) = f(x) + h(x, x_0) \operatorname{int} C$), where $h(x, x_0) = \varepsilon\varphi(x - x_0)$. This proves statement (a) of the proposition.
For the second statement, suppose that $G \in \overline{\mathcal{H}}$ and $C = G$. Then, by definition, x_0 is a (C, h)-quasi proper efficient solution of $(\mathcal{P})$ if there exists $D' \in \mathcal{G}(C)$ such that there is no $x \in S\backslash\{x_0\}$ satisfying that

$$f(x_0) \in f(x) + h(x, x_0)(C + \operatorname{int} D'). \tag{7}$$

By the proof of part (a), statement (7) is equivalent to

$$f(x_0) \in f(x) + \varepsilon\varphi(x - x_0)G + \operatorname{int} D'.$$

Thus, by considering $K := \operatorname{int} D' \cup \{0\}$, it follows that, equivalently, x_0 is a proper generalized ε-quasi minimizer of $(\mathcal{P})$ with respect to (G, φ), and the proof is finished.

Remark 2 In multiobjective optimization, it is specially interesting the case when the ordering cone is polyhedral, since it can be expressed in terms of a matrix and it is easier to manage computationally speaking. In this case, D is defined as

$$D = \{y \in \mathbb{R}^n : Ay \in \mathbb{R}^p_+\}, \tag{8}$$

where $A \in \mathcal{M}_{p\times n}$ (it is a matrix with p rows and n columns), $p \geq n$. We assume that A has no zero rows and that it has full rank, which ensures that D is pointed.
In [27], Kaliszewski introduced the family of closed convex and pointed cones $\{D_\rho\}_{\rho\geq 0}$, where

$$D_\rho := \{y \in \mathbb{R}^n : Ay + \rho UAy \in \mathbb{R}^p_+\}, \tag{9}$$

being $U \in \mathcal{M}_{p\times p}$ the all-ones matrix. It follows that $D_0 = D$, and $D_\rho\backslash\{0\} \subset \operatorname{int} D_{\rho'}$, for all $0 \leq \rho < \rho'$. Hence $D_\rho \in \mathcal{G}(D\backslash\{0\})$, for all $\rho > 0$, i.e., $\{D_\rho\}_{\rho>0}$ is a family of dilating cones for D.

By [27, Lemma 3.7] we have that for any $D' \in \mathcal{G}(D\backslash\{0\})$, there exists $\rho > 0$ such that $D_\rho\backslash\{0\} \subset \operatorname{int} D'$. Therefore, taking into account this property, we can determine quasi proper efficient solutions in terms of the family of cones $\{D_\rho\}_{\rho>0}$. Indeed, by Proposition 1 it follows that

$$\mathrm{QPE}(f, S, C, h) = \bigcup_{\substack{\rho>0 \\ D_\rho\in\mathcal{G}(C)}} \mathrm{QE}(f, S, C + \operatorname{int} D_\rho, h).$$

For this particular setting, the most important approximation sets C used to determine approximate/quasi solutions seem to be $C := q + D$, and $C =$

$q + \operatorname{int} D_\rho$ for $q \in D\backslash\{0\}$, $\rho \geq 0$ (we assume D is solid if $\rho = 0$), due to their easy construction and good properties. These sets C are coradiant sets. In particular, in Theorem 3 we show that by using them, the set of (C, h)-quasi solutions suitably approximates exact solutions of $(\mathcal{P})$.

Proposition 3 *Let $C \in \mathcal{H}$. The following statements hold.*

(a) If C is coradiant, then $\mathrm{QE}(f, S, C, h) \subset \bigcap_{\varepsilon>1} \mathrm{QE}(f, S, C, \varepsilon h)$. Equality holds if, in addition, C is open.
(b) If C is solid, then $\mathrm{QWE}(f, S, C, h) \supset \bigcap_{\varepsilon>1} \mathrm{QWE}(f, S, C, \varepsilon h)$. Equality holds if, in addition, C is coradiant.
(c) If $C \in \overline{\mathcal{H}}$ is coradiant, then

$$\mathrm{QPE}(f, S, C, h) = \bigcup_{D' \in \mathcal{G}(\mathcal{C})} \bigcap_{\varepsilon>1} \mathrm{QWE}(f, S, C + \operatorname{int} D', \varepsilon h).$$

Proof (a) Let $\varepsilon > 1$ and $x_0 \in S$. Since C is coradiant, one has $\varepsilon C \subset C$. Hence for $x \in S\backslash\{x_0\}$, relation $f(x_0) \in f(x) + \varepsilon h(x, x_0)C$ implies $f(x_0) \in f(x) + h(x, x_0)C$. In other words, if x_0 is not $(C, \varepsilon h)$-quasi efficient solution, then it is not (C, h)-quasi efficient solution. This proves the inclusion of (a). Assume C is open. If x_0 is not (C, h)-quasi efficient solution, that is, $f(x_0) \in f(x) + h(x, x_0)C$ for some $x \in S\backslash\{x_0\}$, then for $\varepsilon > 1$ sufficiently close to 1, one still has $f(x_0) \in f(x) + \varepsilon h(x, x_0)C$. Hence x_0 is not $(C, \varepsilon h)$-quasi efficient solution. This proves the second part of (a).
(b) The proof of the second part of (a) reveals that when C is open the inclusion $\bigcap_{\varepsilon>1} \mathrm{QE}(f, S, C, \varepsilon h) \subset \mathrm{QE}(f, S, C, h)$ is true without C being coradiant. This proves the inclusion in (b) due to (1).
Equality in (b) follows directly from (a) and the first part of (b).
(c) By Proposition 1 we have that

$$\mathrm{QPE}(f, S, C, h) = \bigcup_{D' \in \mathcal{G}(C)} \mathrm{QWE}(f, S, C + \operatorname{int} D', h). \tag{10}$$

Since $C + \operatorname{int} D'$ is solid and coradiant, by part (b) we know that

$$\mathrm{QWE}(f, S, C + \operatorname{int} D', h) = \bigcap_{\varepsilon>1} \mathrm{QWE}(f, S, C + \operatorname{int} D', \varepsilon h). \tag{11}$$

Then, part (c) is proved by taking into account (10) and (11).

Remark 3 By means of Proposition 2, we observe that Proposition 3 extends [20, Proposition 3.2] for a general set C, and also improves it in the sense that C does not need to be included in D for the converse inclusion, in contrast to the latter result, in which G is assumed to be contained in D (which implies in particular that $C = G + D\backslash\{0\} \subset D$ is coradiant).

Taking into account Remark 1, (C, h)-quasi solutions behave as approximate solutions where the error is measured by function h and depends, in this way, on the decision variable, in contrast to the notions of (C, ε)-efficiency, where the error is fixed by constant ε. Thus, the (C,h)-quasi solutions can be considered as a type of local approximate solutions.

In the next result, we provide a characterization of (C,h)-quasi solutions in terms of (C,ε)-solutions that points out the type of local nature of quasi solutions. For $x_0 \in S$ and $\varepsilon > 0$, we denote

$$N_h(x_0,\varepsilon) := \{x \in S : h(x,x_0) \leq \varepsilon\}.$$

Theorem 1 *Let $C \in \mathcal{H}$ and $x_0 \in S$. We assume that $h(x,x) = 0$, for all $x \in \mathbb{R}^m$. The following statements hold.*

(a) If C is coradiant, then

$$x_0 \in \mathrm{QE}(f,S,C,h) \Longleftrightarrow x_0 \in \bigcap_{\varepsilon>0} \mathrm{AE}(f,N_h(x_0,\varepsilon),C,\varepsilon).$$

(b) If C is coradiant and $\operatorname{int} C \neq \emptyset$*, then*

$$x_0 \in \mathrm{QWE}(f,S,C,h) \Longleftrightarrow x_0 \in \bigcap_{\varepsilon>0} \mathrm{WAE}(f,N_h(x_0,\varepsilon),C,\varepsilon).$$

(c) If $C \in \overline{\mathcal{H}}$ is coradiant, then

$$\begin{aligned} &x_0 \in \mathrm{QPE}(f,S,C,h) \Longleftrightarrow \\ &\exists D' \in \mathcal{G}(C) \ \textit{ such that } x_0 \in \bigcap_{\varepsilon>0} \mathrm{WAE}(f,N_h(x_0,\varepsilon),C+\operatorname{int} D',\varepsilon). \end{aligned}$$

Proof (a) Let $x_0 \in \mathrm{QE}(f,S,C,h)$. Suppose to the contrary that for some $\bar\varepsilon > 0$, $x_0 \notin \mathrm{AE}(f,N_h(x_0,\bar\varepsilon),C,\bar\varepsilon)$. By asumption on h, we have $x_0 \in N_h(x_0,\bar\varepsilon)$. Then there exists $\bar x \in S \setminus \{x_0\}$ with $h(\bar x,x_0) \leq \bar\varepsilon$ such that $f(x_0) \in f(\bar x) + \bar\varepsilon C$. Set $\delta = \bar\varepsilon/h(\bar x,x_0) \geq 1$. Since C is coradiant, we have $\bar\varepsilon C = h(\bar x,x_0)\delta C \subset h(\bar x,x_0)C$ and then $f(x_0) \in f(\bar x)+h(\bar x,x_0)C$, which contradicts the hypothesis. Conversely, if $x_0 \notin \mathrm{QE}(f,S,C,h)$, then there is some $\bar x \in S \setminus \{x_0\}$ such that $f(x_0) \in f(\bar x) + h(\bar x,x_0)C$. Set $\bar\varepsilon = h(\bar x,x_0) > 0$. Then $\bar x \in N_h(x_0,\bar\varepsilon)$ and $f(x_0) \in f(\bar x) + \bar\varepsilon C$. This implies that $x_0 \notin \mathrm{AE}(f,N_h(x_0,\bar\varepsilon),C,\bar\varepsilon)$, and so $x_0 \notin \bigcap_{\varepsilon>0} \mathrm{AE}(f,N_h(x_0,\varepsilon),C,\varepsilon)$.

Part (b) follows by reasoning in analogous way as in (a). For (c), by Proposition 1 we have that $x_0 \in \mathrm{QPE}(f,S,C,h)$ if and only if there exists $D' \in \mathcal{G}(C)$ such that $x_0 \in \mathrm{QWE}(f,S,C+\operatorname{int} D',h)$. Since $C+\operatorname{int} D'$ is solid and coradiant, by part (b) we know that $x_0 \in \mathrm{QWE}(f,S,C+\operatorname{int} D',h)$ if and only if $x_0 \in \bigcap_{\varepsilon>0} \mathrm{WAE}(f,N_h(x_0,\varepsilon),C+\operatorname{int} D',\varepsilon)$, and the proof is complete.

Remark 4 (a) For $h(x,z) = \|x-z\|$ and $S = \mathbb{R}^m$, the set $N_h(x_0,\varepsilon)$ is the closed ball with center at x_0 and radius ε. For instance, in this case, Theorem 1(a) says that $x_0 \in \mathrm{QE}(f,\mathbb{R}^m,C,h)$ if and only if x_0 is a (C,ε)-efficient solution on the ball with center x_0 and radius ε, for every $\varepsilon > 0$. Hence, the computation of quasi efficient solutions is useful when one is interested in obtaining local (not only global) approximate solutions with a fixed error ε. Indeed, for $h(x,z) =$

$\|x-z\|$ and $S=\mathbb{R}^m$, we have that if x_0 is a quasi efficient solution of $(\mathcal{P})$, then it is a local approximate solution of $(\mathcal{P})$ with precision $\bar{\varepsilon}$ in the ball with center x_0 and radius $\bar{\varepsilon}$.

(b) Taking into account Proposition 1(iii) and Remark 1(b), we know that set $\mathrm{WAE}(f,N_h(x_0,\varepsilon),C+\operatorname{int}D',\varepsilon)$ coincides with $\mathrm{AE}(f,N_h(x_0,\varepsilon),C+\operatorname{int}D',\varepsilon)$ and $\mathrm{He}(f,N_h(x_0,\varepsilon),C+\operatorname{int}D',\varepsilon)$, for all $D'\in\mathcal{G}(C)$. Hence, it follows that the intersection in part (c) of Theorem 1 coincides also with the sets $\bigcap_{\varepsilon>0}\mathrm{He}(f,N_h(x_0,\varepsilon),C+\operatorname{int}D',\varepsilon)$ and $\bigcap_{\varepsilon>0}\mathrm{AE}(f,N_h(x_0,\varepsilon),C+\operatorname{int}D',\varepsilon)$.

Quasi efficient solutions are related to exact efficient solutions of $(\mathcal{P})$ in the following way.

Theorem 2 *Let $C\subset D\backslash\{0\}$ be a coradiant set, and let $h_n:\mathbb{R}^m\times\mathbb{R}^m\to\mathbb{R}_+$, $n\in\mathbb{N}$, with $h_n(x,z)\geq 0$, for all $x,z\in\mathbb{R}^m$, $h_n(x,z)>0$, whenever $x\neq z$, such that h_n pointwisely converges to zero. The following statements hold.*

(a) If $\operatorname{cone}C=D$, then $\displaystyle\bigcap_{n\in\mathbb{N}}\mathrm{QE}(f,S,C,h_n)=\mathrm{E}(f,S,D)$.

(b) If C is solid and $\operatorname{int}D\subset\operatorname{cone}C$, then $\displaystyle\bigcap_{n\in\mathbb{N}}\mathrm{QWE}(f,S,C,h_n)=\mathrm{WE}(f,S,D)$.

(c) If $\operatorname{cone}C=D$, then $\displaystyle\bigcup_{D'\in\mathcal{G}(C)}\bigcap_{n\in\mathbb{N}}\mathrm{QPE}(f,S,C+\operatorname{int}D',h_n)=\mathrm{He}(f,S,D)$.

Proof (a) Take $x_0\in\mathrm{E}(f,S,D)$. Then, $f(x)-f(x_0)\notin -D\backslash\{0\}$, for all $x\in S$. Let $n\in\mathbb{N}$. As $C\subset D\backslash\{0\}$, in particular, we deduce that

$$f(x)+h_n(x,x_0)c-f(x_0)\neq 0,\ \forall x\in S\backslash\{x_0\},\forall c\in C,$$

so $x_0\in\mathrm{QE}(f,S,C,h_n)$. Since that holds for all n, inclusion "$\supset$" is proved. For the converse inclusion, let $x_0\in\displaystyle\bigcap_{n\in\mathbb{N}}\mathrm{QE}(f,S,C,h_n)$. Suppose by reasoning to the contrary that $x_0\notin\mathrm{E}(f,S,D)$. There exists $\bar{x}\in S\backslash\{x_0\}$ such that $f(\bar{x})-f(x_0)=:-\bar{d}\in -D\backslash\{0\}$. Since $\operatorname{cone}C=D$, there is some $t>0$ and $c\in C$ such that $f(\bar{x})-f(x_0)=-\bar{d}=-tc$. Choose n sufficiently large so that $h_n(\bar{x},x_0)\leq t$ and set $\bar{c}=(t/h_n(\bar{x},x_0))c$. Because C is coradiant, $\bar{c}\in C$. We deduce $f(x_0)=f(\bar{x})+tc=f(\bar{x})+h_n(\bar{x},x_0)\bar{c}$, which is a contradiction. Thus, "$\subset$" is also proved.
The proof of (b) is similar to (a). For (c), let $x_0\in\mathrm{He}(f,S,D)$. Then, there exists $D'\in\mathcal{G}(D\backslash\{0\})$ such that $f(x)-f(x_0)\notin-\operatorname{int}D'$, for all $x\in S$. Since $C\subset\operatorname{int}D'$, we have in particular that $D'\in\mathcal{G}(C)$, and as $C+\operatorname{int}D'\subset\operatorname{int}D'$ and $C+\operatorname{int}D'=(C+\operatorname{int}D')+\operatorname{int}D'$ we deduce that for every $n\in\mathbb{N}$

$$f(x)+h_n(x,x_0)c-f(x_0)\neq 0,\ \forall x\in S\backslash\{x_0\},\ \forall c\in(C+\operatorname{int}D')+\operatorname{int}D',$$

so $x_0\in\mathrm{QPE}(f,S,C+\operatorname{int}D',h_n)$, for all $n\in\mathbb{N}$, and inclusion "$\supset$" is proved.

Reciprocally, let $x_0\in\displaystyle\bigcup_{D'\in\mathcal{G}(C)}\bigcap_{n\in\mathbb{N}}\mathrm{QPE}(f,S,C+\operatorname{int}D',h_n)$. Thus, there exists $D'\in\mathcal{G}(C)$ such that $x_0\in\mathrm{QPE}(f,S,C+\operatorname{int}D',h_n)=\mathrm{QE}(f,S,C+$

$\operatorname{int} D', h_n)$, for all $n \in \mathbb{N}$ (see Proposition 1(iii)). Hence, for all $n \in \mathbb{N}$ we have

$$f(x) + h_n(x, x_0)(c + d') - f(x_0) \neq 0, \ \forall x \in S \backslash \{x_0\}, \forall c \in C, \forall d' \in \operatorname{int} D' \quad (12)$$

If $x_0 \notin \mathrm{He}(f, S, D)$ then, since $D \backslash \{0\} + \operatorname{int} D' = \operatorname{int} D'$, and $\operatorname{cone} C = D$, there exist $\bar{x} \in S \backslash \{x_0\}$, $\bar{c} \in \operatorname{cone} C \backslash \{0\}$ and $\bar{d} \in \operatorname{int} D'$ such that $f(\bar{x}) - f(x_0) = -\bar{c} - \bar{d}$. Since C is coradiant and h_n tends to zero pointwise, there exist $\hat{c} \in C$ and $\bar{n} \in \mathbb{N}$ large enough such that $\bar{c} = h_{\bar{n}}(\bar{x}, x_0)\hat{c}$. Hence, by (12), for $x = \bar{x}$, $c = \hat{c}$ and $d' := \frac{\bar{d}}{h_{\bar{n}}(\bar{x}, x_0)} \in \operatorname{int} D'$, we have that

$$-h_{\bar{n}}(\bar{x}, x_0)\hat{c} - \bar{d} + h_{\bar{n}}(\bar{x}, x_0)(\hat{c} + d') \neq 0,$$

a contradiction. So $x_0 \in \mathrm{He}(f, S, D)$, and the proof of (c) is complete.

The following example illustrates the results and improves the understandability of the notion of approximate quasi efficiency.

Example 1 (a) Let $f : \mathbb{R} \to \mathbb{R}^2$ be defined as $f(x) = (x, ||x| - 1|)$, $S = \mathbb{R}$, $D = \mathbb{R}^2_+$, $q = \left(\frac{1}{2}, 1\right)$, $C = q + \mathbb{R}^2_+$ and $h(x, z) = |x - z|$, for all $x, z \in \mathbb{R}$. It is clear that $\mathrm{E}(f, \mathbb{R}, D) = \mathrm{He}(f, \mathbb{R}, D) = (-\infty, -1]$. By the proof of inclusion "$\supset$" in Theorem 2(a),(c) we know that $\mathrm{E}(f, \mathbb{R}, D) \subset \mathrm{QE}(f, \mathbb{R}, C, h)$, and $\mathrm{E}(f, \mathbb{R}, D) \subset \mathrm{QPE}(f, \mathbb{R}, C, h)$. On the other hand, using Theorem 1(a),(c) it is not difficult to deduce that $\mathrm{QE}(f, \mathbb{R}, C, h) = \mathrm{QPE}(f, \mathbb{R}, C, h) = \mathrm{E}(f, \mathbb{R}, D) \cup (0, 1]$.

(b) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined as $f(x_1, x_2) = \left(x_1, x_2^2\right)$, $S = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = |x_1| - 1, |x_1| \leq 2\}$, $D = \mathbb{R}^2_+$, $q = \left(1, \frac{1}{2}\right)$, $C = q + \mathbb{R}^2_+$ and $h_n(x, z) = \frac{1}{n}\|x - z\|_\infty$ for all $x, z \in \mathbb{R}^2$, where $\|(x_1, x_2)\|_\infty = \max\{|x_1|, |x_2|\}$ and $n \geq 1$ is given. It is clear that $\mathrm{E}(f, S, D) = \{(t_1, t_2) \in S : -2 \leq t_1 \leq -1\}$. We compute $\mathrm{QE}(f, S, C, h_n)$. Let $x_0 = (t_1, t_2) \in S$ and $x = (x_1, x_2) \in S$. By definition, $x_0 \in \mathrm{QE}(f, \mathbb{R}, C, h)$ if and only if the inclusion

$$f(x) - f(x_0) \in -\frac{1}{n}\|x - x_0\|_\infty q - \mathbb{R}^2_+ \quad (13)$$

has no solution $x \in S \setminus \{x_0\}$. Observe that if x is a solution of (13), then $f(x) \in f(x_0) - \operatorname{int} \mathbb{R}^2_+$, which implies that $x_1 < t_1$. Hence, $\|x - x_0\|_\infty = t_1 - x_1$. Inclusion (13) becomes the system

$$\begin{aligned} x_1 - t_1 &\leq -(t_1 - x_1)/n \\ x_2^2 - t_2^2 &\leq -(t_1 - x_1)/(2n). \end{aligned} \quad (14)$$

Since the first inequality holds for all $x_1, t_1 \in S$ satisfying $t_1 > x_1$, we obtain an equivalent system for (14):

$$\begin{aligned} t_1 + x_1 \geq 2 + 1/(2n) &\ \text{ for } \ x_1 \in [0, 2], \\ t_1 + x_1 \geq -2 + 1/(2n) &\ \text{ for } \ x_1, t_1 \in [-2, 0), \\ t_1^2 - (2 + 1/(2n))t_1 \geq x_1^2 + (2 - 1/(2n))x_1 &\ \text{ for } \ x_1 \in [-2, 0), t_1 \in [0, 2]. \end{aligned}$$

It follows from these inequalities that $\mathrm{QE}(f, S, C, h_n) = \{(t_1, t_2) \in S : t_1 \in [-2, -1 + 1/(4n)) \cup (1 + 1/(4n) - 1/\sqrt{n}, 1 + 1/(4n))\}$. For $n \geq 1$, the inclusions $\mathrm{E}(f, S, D) \subset \mathrm{QE}(f, S, C, h_n) \subset \mathrm{QE}(f, S, C, h_{n+1})$ are strict. Moreover,

$\bigcap_{n\in\mathbb{N}} \mathrm{QE}(f,S,C,h_n) = \{(t_1,t_2) \in S : t_1 \in [-2,-1] \cup \{1\}\} \neq \mathrm{E}(f,S,D)$. Notice that the hypothesis of Theorem 2 (a) is not satisfied because $\operatorname{cone} C \neq D$. By a similar argument one can see that $\mathrm{QWE}(f,S,C,h_n) = \{(t_1,t_2) \in S : t_1 \in [-2,-1+1/(4n)] \cup [1+1/(4n)-1/\sqrt{n}, 1+1/(4n)]\}$, for $n \geq 2$, and $\bigcap_{n\in\mathbb{N}} \mathrm{QWE}(f,S,C,h_n) = \{(t_1,t_2) \in S : t_1 \in [-2,-1] \cup \{1\} = \mathrm{WE}(f,S,D)$.

To finish this section, in the following result we show that quasi solutions with respect to the coradiant sets $C = q + D$ and $C = q + \operatorname{int} D'$, $D' \in \mathcal{G}(D\backslash\{0\})$, $q \in D\backslash\{0\}$, are useful to approximate exact solutions of problem $(\mathcal{P})$.

Theorem 3 *Let $x_0 \in S$, $q \in D\backslash\{0\}$, $h_n : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}_+$, $n \in \mathbb{N}$, with $h_n(x,z) \geq 0$, for all $x,z \in \mathbb{R}^m$, $h_n(x,z) > 0$, whenever $x \neq z$, and let $(x_n) \subset \mathbb{R}^m$ be a sequence such that $x_n \in \mathrm{QE}(f,S,q+\operatorname{int}\hat{D},h_n)$, for all $n \in \mathbb{N}$, $f(x_n) \to f(x_0)$, and $h_n(\cdot,x_n) \to 0$, where $\hat{D} \in \mathcal{G}(D\backslash\{0\}) \cup \{D\}$ (we assume D is solid when $\hat{D} = D$).*

(a) If $\hat{D} \in \mathcal{G}(D\backslash\{0\})$, then $x_0 \in \mathrm{He}(f,S,D)$.
(b) If $\hat{D} = D$, then $x_0 \in \mathrm{WE}(f,S,D)$.

Proof (a) We have that $C := q + \operatorname{int}\hat{D}$ is open and coradiant and $\operatorname{cone} C = \operatorname{int}\hat{D}$. We affirm that $x_0 \in \mathrm{WE}(f,S,\hat{D})$. Indeed, if not, then there exists $\bar{x} \in S$ such that $f(\bar{x}) - f(x_0) \in -\operatorname{int}\hat{D}$, so there exists $\alpha > 0$ such that $f(\bar{x}) - f(x_0) \in -\alpha C$, and as αC is open and $f(x_n) \to f(x_0)$, there exists $n_0 \in \mathbb{N}$ such that

$$f(\bar{x}) - f(x_n) \in -\alpha C,\ \forall n \geq n_0. \tag{15}$$

On the other hand, as $h_n(\bar{x},x_n) \to 0$, there exists $n_1 \in \mathbb{N}$ verifying that $h_n(\bar{x},x_n) \leq \alpha$, for all $n \geq n_1$. Since C is coradiant, we have $h_n(\bar{x},x_n)C \supset \alpha C$, for all $n \geq n_1$, and then by (15) it follows that

$$f(\bar{x}) - f(x_n) \in -h_n(\bar{x},x_n)C,\ \forall n \geq \max\{n_0,n_1\},$$

which contradicts that $x_n \in \mathrm{QE}(f,S,q+\operatorname{int}\hat{D},h_n)$, for all $n \in \mathbb{N}$. Thus, $x_0 \in \mathrm{WE}(f,S,\hat{D})$, which implies by definition that $x_0 \in \mathrm{He}(f,S,D)$.

(b) It is clear that $x_n \in \mathrm{QE}(f,S,q+D,h_n) \subset \mathrm{QE}(f,S,q+\operatorname{int} D,h_n)$, for all $n \in \mathbb{N}$. Let $C := q + \operatorname{int} D$. The set C is open, coradiant and $\operatorname{cone} C \setminus \{0\} = \operatorname{int} D$. Thus, by reasoning in analogous way as in part (a) for this set C, we conclude that $x_0 \in \mathrm{WE}(f,S,D)$, and the proof is complete.

Remark 5 (a) Under the assumptions of Theorem 3, let us observe that by Proposition 1, we have that $x_n \in \mathrm{QPE}(f,S,q+\operatorname{int}\hat{D},h_n) \subset \mathrm{QPE}(f,S,q+D,h_n)$, for all $\hat{D} \in \mathcal{G}(D\backslash\{0\})$, $n \in \mathbb{N}$.

(b) In particular, if D is polyhedral, then in Theorem 3 we can deal with cones D_ρ, $\rho > 0$ (see (9)), to reach exact proper solutions. In this way, the set $C = q + \operatorname{int} D_\rho$ is easy to construct.

4 Scalarization results

In this section we are going to provide characterizations for (C,h)-quasi weak and proper solutions through linear and nonlinear scalarization. These conditions generalize several known optimality results obtained via scalarization approaches for well-known notions of exact and approximate efficient solutions.

Let $\alpha \geq 0$ and $\varphi : \mathbb{R}^n \to \overline{\mathbb{R}}$. We introduce the following notations

$$\mathrm{SAE}_S(f,\varphi,\alpha) = \{x_0 \in S : 0 \leq \varphi(f(x) - f(x_0)) + \alpha h(x,x_0),\ \forall x \in S\},$$
$$\mathrm{SSAE}_S(f,\varphi,\alpha) = \{x_0 \in S : 0 < \varphi(f(x) - f(x_0)) + \alpha h(x,x_0),\ \forall x \in S\backslash\{x_0\}\},$$

and $\kappa_E(\varphi) = \inf_{e\in E} \varphi(e)$, for $\emptyset \neq E \subset \mathbb{R}^n\backslash\{0\}$.

4.1 Linear scalarization

In the following theorems we derive characterizations for (C,h)-quasi solutions through linear scalarization. For the first one, no convexity assumptions are required.

Theorem 4 *Let $C \in \mathcal{H}$. The following statements hold.*

(a) $\bigcup_{\xi\in C^+\backslash\{0\}} \mathrm{SSAE}_S(f,\xi,\kappa_C(\xi)) \subset \mathrm{QE}(f,S,C,h)$.
(b) $\bigcup_{\xi\in C^+\backslash\{0\}} \mathrm{SAE}_S(f,\xi,\kappa_C(\xi)) \subset \mathrm{QWE}(f,S,C,h)$ *if* $\operatorname{int} C \neq \emptyset$.
(c) $\bigcup_{\xi\in D^{s+}\cap C^+} \mathrm{SAE}_S(f,\xi,\kappa_C(\xi)) \subset \mathrm{QPE}(f,S,C,h)$ *if* $C \in \overline{\mathcal{H}}$.

Proof (a) Let $\xi \in C^+\backslash\{0\}$ and $x_0 \in \mathrm{SSAE}_S(f,\xi,\kappa_C(\xi))$ and suppose by reasoning to the contrary that $x_0 \notin \mathrm{QE}(f,S,C,h)$. Then, there exist $\bar{x} \in S\backslash\{x_0\}$ and $\bar{c} \in C$ such that $f(\bar{x}) + h(\bar{x},x_0)\bar{c} - f(x_0) = 0$. Hence,

$$(\xi \circ f)(x_0) = (\xi \circ f)(\bar{x}) + h(\bar{x},x_0)\langle\xi,\bar{c}\rangle \geq (\xi \circ f)(\bar{x}) + h(\bar{x},x_0)\kappa_C(\xi), \quad (16)$$

and we reach a contradiction. Therefore, $x_0 \in \mathrm{QE}(f,S,C,h)$.

The proof of (b) follows with the same reasoning as for (a), taking into account that $\langle\xi,c\rangle > \kappa_C(\xi)$ for all $c \in \operatorname{int} C$.

(c) Let $\xi \in D^{s+} \cap C^+$ and $x_0 \in \mathrm{SAE}_S(f,\xi,\kappa_C(\xi))$, and let $D' := \{y \in \mathbb{R}^n : \langle\xi,y\rangle > 0\} \cup \{0\}$. It is clear that $D' \in \mathcal{G}(C)$. Again, suppose by reasoning to the contrary that $x_0 \notin \mathrm{QPE}(f,S,C,h)$. Then, in particular, there exist $\bar{x} \in S\backslash\{x_0\}$ and $\bar{c} \in C$ such that $f(\bar{x}) + h(\bar{x},x_0)\bar{c} - f(x_0) \in -\operatorname{int} D'$, so

$$(\xi \circ f)(x_0) > (\xi \circ f)(\bar{x}) + h(\bar{x},x_0)\langle\xi,\bar{c}\rangle \geq (\xi \circ f)(\bar{x}) + h(\bar{x},x_0)\kappa_C(\xi),$$

a contradiction. Then, $x_0 \in \mathrm{QPE}(f,S,C,h)$.

Remark 6 (a) Theorem 4(a)-(b) extends [20, Proposition 4.1] for generalized ε-quasi minimizers of problem $(\mathcal{P})$ with respect to the pair (G,φ). Indeed, the set of (C,h)-quasi efficient solutions is equal to the set of generalized ε-quasi minimizers with respect to (G,φ), when $C = G + D\backslash\{0\}$ and $h(x,z) = \varepsilon\varphi(x-z)$, $\varepsilon > 0$ (see Proposition 2) and observe that

$$C^+\backslash\{0\} = G^+ \cap D^+\backslash\{0\} \supset P \cap D^+\backslash\{0\} \supset P \cap D^{s+},$$

for $P = \{\xi \in \mathbb{R}^n : \kappa_G(\xi) > 0\} \cup \{0\}$ (see [20, Proposition 4.1]). Moreover, for $C = G + D\backslash\{0\}$ and $\xi \in P \cap D^{s+}$, set $\mathrm{SSAE}_S(f, \xi, \kappa_C(\xi))$ in Theorem 4(a) can be replaced by $\mathrm{SAE}_S(f, \xi, \kappa_C(\xi))$. This is because in the proof of Theorem 4(a), element $\bar{c}$ can be expressed as $\bar{g} + \bar{d}$, for some $\bar{g} \in G$ and $\bar{d} \in D\backslash\{0\}$. Then, statement (16) takes the form

$$\begin{aligned}(\xi \circ f)(x_0) &= (\xi \circ f)(\bar{x}) + h(\bar{x}, x_0)\langle \xi, \bar{g} + \bar{d}\rangle \\ &\geq (\xi \circ f)(\bar{x}) + h(\bar{x}, x_0)\kappa_G(\xi) + h(\bar{x}, x_0)\langle \xi, \bar{d}\rangle \\ &> (\xi \circ f)(\bar{x}) + h(\bar{x}, x_0)\kappa_C(\xi),\end{aligned}$$

since $\kappa_G(\xi) = \kappa_C(\xi)$, obtaining the corresponding contradiction.

(b) If $h(x, z) = \varepsilon > 0$, for all $x, z \in \mathbb{R}^m$, then Theorem 4(c) reduces to [15, Theorem 4.4].

Under generalized convexity hypotheses, we obtain some scalarization results for (C, h)-quasi solutions for special types of sets C. We remind that a set-valued mapping $F : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is said to be nearly D-subconvexlike on S if $\mathrm{cl}\,\mathrm{cone}(F(S) + D)$ is convex, where $F(S) := \bigcup_{x \in S} F(x)$ (see [43]). For (C, h)-quasi weak efficient solutions we have the following result, the first statement of which deals with a particular case when $C = C + D$ and has been given in [19, Theorem 4.1].

Theorem 5 *Let $C \in \mathcal{H}$ be a coradiant set and $x_0 \in S$. The following statements hold.*

(a) Assume that D is solid, $C + D = C$ and that the mapping $x \to f(x) - f(x_0) + h(x, x_0)C$ is nearly D-subconvexlike on S. If $x_0 \in \mathrm{QWE}(f, S, C, h)$, then there is $\xi \in C^+\backslash\{0\}$ such that $x_0 \in \mathrm{SAE}_S(f, \xi, \kappa_C(\xi))$.

(b) Assume that C is convex, $\mathrm{cone}\, C$ is solid and the mapping $x \to f(x) - f(x_0) + h(x, x_0)C$ is nearly $\mathrm{cone}\, C$-subconvexlike on S. If $x_0 \in \mathrm{QWE}(f, S, C, h)$, then there is $\xi \in C^+\backslash\{0\}$ such that $x_0 \in \mathrm{SAE}_S(f, \xi, \kappa_C(\xi))$.

Proof We prove (b). Since C is convex and coradiant, by [17, Lemma 3.1] we know that $\mathrm{cone}\, C$ is convex and $C + \mathrm{cone}\, C = C$. Thus, it is easy to see that $x_0 \in \mathrm{QWE}(f, S, C, h)$ if and only if

$$f(x) + h(x, x_0)c - f(x_0) \notin -\,\mathrm{int}\,\mathrm{cone}\, C,\ \forall x \in S,\ \forall c \in C,$$

and the latter is equivalent to

$$\mathrm{cl}\,\mathrm{cone}(F(S) + \mathrm{cone}\, C) \cap (-\,\mathrm{int}\,\mathrm{cone}\, C) = \emptyset,$$

where

$$F(x) = f(x) - f(x_0) + h(x, x_0)C. \tag{17}$$

Since both of the sets in the above intersection are convex, by a separation approach we find a nonzero vector $\xi \in (\mathrm{int}\,\mathrm{cone}\, C)^+ = C^+$ such that $\langle \xi, f(x) - f(x_0) + h(x, x_0)c\rangle \geq 0$ for all $x \in S$ and $c \in C$. This implies $\langle \xi, f(x) - f(x_0)\rangle + h(x, x_0)\kappa_C(\xi) \geq 0$ for all $x \in S$, that is $x_0 \in \mathrm{SAE}_S(f, \xi, \kappa_C(\xi))$. The proof is complete.

From Theorems 4(b) and 5 we deduce the following corollary, the first condition of which was given in ([19, Corollary 4.1]).

Corollary 1 *Let $C \in \mathcal{H}$ be coradiant. Then,*

$$\mathrm{QWE}(f, S, C, h) = \bigcup_{\xi \in C^+ \setminus \{0\}} \mathrm{SAE}_S(f, \xi, \kappa_C(\xi)). \tag{18}$$

provided either of the following conditions holds.

(a) D is solid, $C + D = C$ and the mapping $x \to f(x) - f(x_0) + h(x, x_0)C$ is nearly D-subconvexlike on S for all $x_0 \in S$.

(b) C is convex, $\operatorname{cone} C$ is solid and the mapping $x \to f(x) - f(x_0) + h(x, x_0)C$ is nearly $\operatorname{cone} C$-subconvexlike on S for all $x_0 \in S$.

For the (C, h)-quasi proper solutions, we have the following result.

Theorem 6 *Let $C \in \overline{\mathcal{H}}$ be a coradiant set and let $x_0 \in S$. Suppose that the mapping $x \to f(x) - f(x_0) + h(x, x_0)C$ is nearly D-subonvexlike on S. If $x_0 \in \mathrm{QPE}(f, S, C, h)$, then there exists $\xi \in D^{s+} \cap C^+$ such that $x_0 \in \mathrm{SAE}_S(f, \xi, \kappa_C(\xi))$.*

Proof If $x_0 \in \mathrm{QPE}(f, S, C, h)$, then it is easy to see that there exists $D' \in \mathcal{G}(C)$ such that $\operatorname{cl}\operatorname{cone}(F(S) + D) \cap (-\operatorname{int} D') = \emptyset$, where F is given in (17). Then, by applying a similar reasoning as in [19, Theorem 4.1] we deduce that there exists $\xi \in (D'^+ \setminus \{0\}) \cap C^+ \subset D^{s+} \cap C^+$ such that $x_0 \in \mathrm{SAE}_S(f, \xi, \kappa_C(\xi))$, and the proof is complete.

Example 2 Consider the same data of Example 1(a), let $x_0 = \frac{1}{2}$ and define $g : \mathbb{R} \to \mathbb{R}$ as $g(x) = ||x| - 1|$. Then, mapping $x \to F(x) := f(x) - f(x_0) + h(x, x_0)C$ is nearly D-subconvexlike on $\mathbb{R}$. Indeed, this mapping assigns to each $x \in \mathbb{R}$ the set

$$\left(x - \frac{1}{2} + \frac{1}{2}\left|x - \frac{1}{2}\right|, g(x) - \frac{1}{2} + \left|x - \frac{1}{2}\right|\right) + \mathbb{R}^2_+.$$

So $F(\mathbb{R}) + \mathbb{R}^2_+ = \operatorname{Im} c + \mathbb{R}^2_+$, where $\operatorname{Im} c$ denotes the image of the piecewise linear curve $c : \mathbb{R} \to \mathbb{R}^2$ defined as $c(x) = \left(x - \frac{1}{2} + \frac{1}{2}\left|x - \frac{1}{2}\right|, g(x) - \frac{1}{2} + \left|x - \frac{1}{2}\right|\right)$.

It follows that $\operatorname{Im} c + \mathbb{R}^2_+$ is not a convex set, but $\operatorname{cl}\operatorname{cone}(F(\mathbb{R}) + \mathbb{R}^2_+)$ is convex, so by Theorem 6 we deduce that if $x_0 \in \mathrm{QPE}(f, \mathbb{R}, C, h)$ then there exists $\xi \in D^{s+} \cap C^+ = \operatorname{int} \mathbb{R}^2_+$ such that $x_0 \in \mathrm{SAE}_S(f, \xi, \langle \xi, q \rangle)$.

From Theorems 4(c) and 6 we deduce the following characterization of (C, h)-quasi proper efficient solutions.

Corollary 2 *Let $C \in \overline{\mathcal{H}}$ be a coradiant set and suppose that the mapping $x \to f(x) - f(x_0) + h(x, x_0)C$ is nearly D-subonvexlike on S for all $x_0 \in S$. Then,*

$$\mathrm{QPE}(f, S, C, h) = \bigcup_{\xi \in D^{s+} \cap C^+} \mathrm{SAE}_S(f, \xi, \kappa_C(\xi)).$$

Remark 7 (*a*) In [19, Proposition 4.1] it was proved that if f is D-convex on S (i.e., $\lambda f(x_1)+(1-\lambda)f(x_2)\in f(\lambda x_1+(1-\lambda)x_2)+D$, for all $x\in S$), $h(\cdot,x_0)$ is convex on S and $C\subset D$ is convex, then the mapping $x\to f(x)-f(x_0)+h(x,x_0)C$ is nearly D-subconvexlike on S.

(*b*) If $h(x,z)=\varepsilon>0$, for all $x,z\in\mathbb{R}^m$, then Theorem 6 reduces to [15, Theorem 4.5].

Remark 8 Let D be polyhedral (defined as in (8)). it is clear that

$$D^+=\{\xi\in\mathbb{R}^n:\xi=\lambda_1a_1+\lambda_2a_2+\cdots+\lambda_pa_p,\lambda_1,\lambda_2,\ldots,\lambda_p\geq 0\},$$

where a_i denotes the i-th row of the matrix A that defines D. Since D is a closed and pointed cone in $\mathbb{R}^n$, we know that $\operatorname{int}(D^+)\neq\emptyset$ (see, for instance [10]). Hence, by [25, Lemma 3.21(d)] we have that $D^{s+}=\operatorname{int}(D^+)$ and then by [41, Corollary 6.6.2] we deduce that

$$D^{s+}=\{\xi\in\mathbb{R}^n:\xi=\lambda_1a_1+\lambda_2a_2+\cdots+\lambda_pa_p,\lambda_1,\lambda_2,\ldots,\lambda_p>0\}.$$

In this context, by Remark 2 we know that (C,h)-quasi proper efficient solutions can be obtained in terms of the dilating cones D_ρ, $\rho\geq 0$. With the same reasoning as above, taking into account the definition of D_ρ, $\rho\geq 0$, we have that

$$D_\rho^+=\left\{\xi\in\mathbb{R}^n:\xi=\sum_{i=1}^p\lambda_ia_i+\rho\left(\sum_{i=1}^p\lambda_i\right)\left(\sum_{i=1}^p a_i\right):\lambda_1,\lambda_2,\ldots,\lambda_p\geq 0\right\},$$

and D_ρ^{s+} has the same definition but considering that λ_i is strictly positive, for all $i=1,2,\ldots,p$.

We also know by Theorem 3 that quasi proper efficient solutions with respect to the improvement sets $C=q+D$, or $C=q+\operatorname{int}D_\rho$, for $q\in D\backslash\{0\}$, $\rho>0$, suitably approximate exact proper and weak efficient solutions of $(\mathcal{P})$.

In the next corollary we characterize quasi proper efficient solutions through linear scalarization, when D is polyhedral and for these sets C. Taking into account this result and Remark 8, we see that the calculus of these solutions is rather simple from a computational point of view.

Corollary 3 *Assume that D is polyhedral. Let $q\in D\backslash\{0\}$ and $\rho\geq 0$.*

(*a*) *Suppose that $\rho>0$ and the mapping $x\to f(x)-f(x_0)+h(x,x_0)(q+\operatorname{int}D_\rho)$ is nearly D-subconvexlike on S for all $x_0\in S$. Then,*

$$\operatorname{QPE}(f,S,q+\operatorname{int}D_\rho,h)=\bigcup_{\xi\in D_\rho^+\backslash\{0\}}\operatorname{SAE}_S(f,\xi,\langle\xi,q\rangle).$$

(*b*) *Suppose that the mapping $x\to f(x)-f(x_0)+h(x,x_0)(q+D)$ is nearly D-subconvexlike on S for all $x_0\in S$. Then,*

$$\operatorname{QPE}(f,S,q+D,h)=\bigcup_{\xi\in D^{s+}}\operatorname{SAE}_S(f,\xi,\langle\xi,q\rangle),$$

where $\mathrm{SAE}_S(f,\xi,\langle\xi,q\rangle)=\{x_0\in S:\langle\xi,f(x)-f(x_0)+h(x,x_0)q\rangle\geq 0,\ \forall x\in S\}$.

Proof Observe that for $C=q+\operatorname{int}D_\rho$ and $C=q+D$ one has $\kappa_C(\xi)=\langle\xi,q\rangle$ for $\xi\in D_\rho^+\backslash\{0\}$ and $\xi\in D^{s+}$ respectively. It remains to apply Corollary 2 to complete the proof.

Example 3 Consider the same data of Example 1(a) and let $x_0=\frac{1}{2}$. Then, for $\xi=(4,3)\in D^{s+}=\operatorname{int}\mathbb{R}^2_+$ we have that

$$\begin{aligned}\langle\xi,f(x)-f(x_0)+h(x,x_0)q\rangle&=4\left(x-\frac{1}{2}\right)+3\left(||x|-1|-\frac{1}{2}\right)+5\left|x-\frac{1}{2}\right|\\&\geq 0,\ \forall x\in\mathbb{R},\end{aligned}$$

so $x_0\in\mathrm{SAE}_S(f,\xi,\langle\xi,q\rangle)$, and then, by Theorem 4(c), $x_0\in\mathrm{QPE}(f,S,q+\mathbb{R}^2_+,h)$.

4.2 Nonlinear scalarization

To obtain characterizations of quasi efficient solutions through nonlinear scalarization we use the well-known functional defined below (see, for instance [9, 34]). Let $\emptyset\neq H\subset\mathbb{R}^n$ and $q\in\mathbb{R}^n\backslash\{0\}$. The functional $\varphi_{H,q}:\mathbb{R}^n\to\mathbb{R}\cup\{\pm\infty\}$ is defined as

$$\varphi_{H,q}(y)=\inf\{t\in\mathbb{R}:y\in tq-H\},$$

where it is understood that $\varphi_{H,q}(y)=+\infty$ if $\{t\in\mathbb{R}:y\in tq-H\}=\emptyset$.

Remark 9 In particular, for $D'\in\mathcal{G}(D\backslash\{0\})$ and $q\in D\backslash\{0\}$, we know (see, for instance [10, Theorem 2.3.1]) that $\varphi_{D',q}$ is convex (so it is continuous), finite valued, subadditive and positively homogeneous, and for all $\lambda\in\mathbb{R}$ we have $\varphi_{D',q}(y+\lambda q)=\varphi_{D',q}(y)+\lambda$, for all $y\in\mathbb{R}^n$, and

$$\begin{aligned}\{y\in\mathbb{R}^n:\varphi_{D',q}(y)<\lambda\}&=\lambda q-\operatorname{int}D',\\\{y\in\mathbb{R}^n:\varphi_{D',q}(y)\leq\lambda\}&=\lambda q-\operatorname{cl}D'.\end{aligned}$$

In the following two results we establish characterizations for (C,h)-quasi proper solutions, for a special type of sets C.

Theorem 7 *Let $q\in D\backslash\{0\}$ and $C\in\overline{\mathcal{H}}$. Then*

$$\mathrm{QPE}(f,S,C,h)\subset\bigcup_{D'\in\mathcal{G}(C)}\mathrm{SAE}_S(f,\varphi_{D',q},\kappa_C(\varphi_{D',q})).\tag{19}$$

Moreover, if $C=E+\operatorname{int}D'$ for some $D'\in\mathcal{G}(C)$ and $\emptyset\neq E\subset\mathbb{R}^n\setminus\{0\}$, then

$$\mathrm{QPE}(f,S,C,h)\subset\mathrm{SAE}_S(f,\varphi_{D',q},\kappa_E(\varphi_{D',q}));\tag{20}$$

and if additionally $C\subset\lambda q+\operatorname{int}D'$ with $\lambda>0$, then

$$\mathrm{QPE}(f,S,C,h)\supset\mathrm{SAE}_S(f,\varphi_{D',q},\lambda).\tag{21}$$

Proof Let $x_0 \in \mathrm{QPE}(f, S, C, h)$. By definition, there is some $D' \in \mathcal{G}(C)$ such that

$$f(x) + h(x, x_0)c - f(x_0) \notin -\operatorname{int} D', \ \forall x \in S, \ \forall c \in C.$$

By the properties of functional $\varphi_{D',q}$ (see Remark 9) we obtain

$$\begin{aligned} 0 &\leq \varphi_{D',q}(f(x) - f(x_0) + h(x, x_0)c) \\ &\leq \varphi_{D',q}(f(x) - f(x_0)) + \varphi_{D',q}(c)h(x, x_0), \ \forall x \in S, \ \forall c \in C, \end{aligned}$$

which implies that

$$0 \leq \varphi_{D',q}(f(x) - f(x_0)) + \kappa_C(\varphi_{D',q})h(x, x_0), \ \forall x \in S.$$

By definition, $x_0 \in \mathrm{SAE}_S(f, \varphi_{D',q}, \kappa_C(\varphi_{D',q}))$.
For the second part, we observe that in view of Proposition 1(iii),

$$\mathrm{QPE}(f, S, C, h) = \mathrm{QE}(f, S, E + \operatorname{int} D', h).$$

Therefore, for $x_0 \in \mathrm{QPE}(f, S, C, h)$ one has

$$f(x) + h(x, x_0)e - f(x_0) \notin -\operatorname{int} D', \ \forall x \in S, \ \forall e \in E.$$

The argument we employed to prove the first part yields that x_0 belongs to $\mathrm{SAE}_S(f, \varphi_{D',q}, \kappa_E(\varphi_{D',q}))$.
For the last part, let $x_0 \in \mathrm{SAE}_S(f, \varphi_{D',q}, \lambda)$. Suppose to the contrary that $x_0 \notin \mathrm{QPE}(f, S, C, h)$. One can find some $x \in S \backslash \{x_0\}$ such that $f(x_0) \in f(x) + h(x, x_0)(C + \operatorname{int} D') \subset f(x) + h(x, x_0)\lambda q + \operatorname{int} D'$. It follows that $\varphi_{D',q}(f(x) - f(x_0) + \lambda h(x, x_0)q) = \varphi_{D',q}(f(x) - f(x_0)) + \lambda h(x, x_0) < 0$, which is a contradiction. The proof is complete.

From Theorem 7 we deduce the following corollary for specific (C, h)-quasi proper solutions of $(\mathcal{P})$.

Corollary 4 *Let $q \in D \backslash \{0\}$. The following statements hold.*

(a) If $C = q + D$, then

$$\mathrm{QPE}(f, S, C, h) = \bigcup_{D' \in \mathcal{G}(D \backslash \{0\})} \mathrm{SAE}_S(f, \varphi_{D',q}, 1).$$

(b) If $C = q + \operatorname{int} D'$ for some $D' \in \mathcal{G}(D \backslash \{0\})$, then

$$\mathrm{QPE}(f, S, C, h) = \mathrm{SAE}_S(f, \varphi_{D',q}, 1).$$

Proof Inclusion "$\subset$" of part (a) follows by statement (19), and inclusion "$\supset$" is obtained by applying an analogous reasoning as in the proof of the last part of Theorem 7. Part (b) is deduced from (20) and (21).

Remark 10 Let $D' \in \mathcal{G}(D\backslash\{0\})$, $r > 0$ and $q \in D\backslash\{0\}$.

(a) Let define $\mathcal{A}_{D',r} := \{E \in \overline{\mathcal{H}} : E \cap (rq - \operatorname{int} D') = \emptyset\}$, $\mathcal{A}_{D'} = \bigcup_{r>0} \mathcal{A}_{D',r}$. It is easy to see from Remark 9 that $E \in \mathcal{A}_{D',r}$ if and only if $\kappa_E(\varphi_{D',q}) \geq r$, so if the set E of Theorem 7 belongs to $\mathcal{A}_{D'}$, then $\kappa_E(\varphi_{D',q}) > 0$. For instance, if $0 \notin \operatorname{cl}(E + D')$ then $E \in \mathcal{A}_{D'}$ (see [21, Remark 6]). This latter condition holds for example when E is compact, $E \cap (-D') = \emptyset$.

(b) In Theorem 7, it is clear that if $E \subset q + \operatorname{int} D'$, then $E \in \mathcal{A}_{D'}$. In particular, if $E \subset D\backslash\{0\}$ is compact and $k_0 \in \mathbb{R}^n\backslash\{0\}$, then there exists $r > 0$ such that $E \subset rk_0 + \operatorname{int} D'$. Indeed, if not, for every $n \in \mathbb{N}$ there exists $e_n \in E$ such that $e_n - \frac{1}{n}k_0 \in (\operatorname{int} D')^c$. Since E is compact, we can suppose without loss of generality that $e_n \to \bar{e} \in E$, so taking the limit we deduce that $\bar{e} \in (\operatorname{int} D')^c$, which is a contradiction, since $E \subset \operatorname{int} D'$. Thus, for $E \subset D\backslash\{0\}$ compact we can find $q \in D\backslash\{0\}$ such that $E \subset q + \operatorname{int} D'$, in particular, $C = E + \operatorname{int} D' \subset q + \operatorname{int} D'$.

Remark 11 Suppose that D is polyhedral and let $q \in D\backslash\{0\}$ and $\rho > 0$. By [16, Lemma 2.3] we have that

$$\varphi_{D_\rho,q}(y) = \max_{i\in\{1,2,\ldots,p\}} \frac{\langle \alpha_i, y\rangle}{\langle \alpha_i, q\rangle}, \text{ for all } y \in \mathbb{R}^n,$$

where $\alpha_i = a_i + \rho \sum_{j=1}^{p} a_j$, $i \in \{1, 2, \ldots, p\}$.

Hence, by Remarks 2 and 11, Corollary 4 reduces to the following one, when D is polyhedral.

Corollary 5 *Suppose that D is polyhedral and let $q \in D\backslash\{0\}$ and $\rho > 0$. The following statements hold.*

(a) If $C = q + D$, then

$$\operatorname{QPE}(f, S, C, h) = \bigcup_{\rho>0} \operatorname{SAE}_S(f, \varphi_{D_\rho,q}, 1),$$

(b) If $C = q + \operatorname{int} D_\rho$, then

$$\operatorname{QPE}(f, S, C, h) = \operatorname{SAE}_S(f, \varphi_{D_\rho,q}, 1),$$

where

$$\operatorname{SAE}_S(f, \varphi_{D_\rho,q}, 1) = \left\{ x_0 \in S : \max_{i\in\{1,2,\ldots,p\}} \frac{\langle \alpha_i, f(x) - f(x_0)\rangle}{\langle \alpha_i, q\rangle} \geq -h(x, x_0),\ \forall x \in S \right\}.$$

5 A unified notion of subdifferential for vector mappings

In this section, we use quasi efficient solutions to extend the notion of weak subdifferential given in Definition 1 to the vector case which helps us to derive several subdifferentials known in vector optimization for vector mappings in a unifying way. In what follows, $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$ stands for the set of all linear mappings from $\mathbb{R}^m$ to $\mathbb{R}^n$.

Definition 5 Let $x_0 \in \operatorname{dom} f$. We define the following subdifferentials of f at x_0:

(a) Let $C \in \mathcal{H}$. The (C, h)-subdifferential of f at x_0 is defined as

$$\partial^{\mathrm{E}}_{C,h} f(x_0) := \{T \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n) : x_0 \in \mathrm{QE}(f - T, \operatorname{dom} f, C, h)\}.$$

(b) Let $C \in \mathcal{H}$. The weak (C, h)-subdifferential of f at x_0 is defined as

$$\partial^{\mathrm{WE}}_{C,h} f(x_0) := \{T \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n) : x_0 \in \mathrm{QWE}(f - T, \operatorname{dom} f, C, h)\}.$$

(c) Let $C \in \overline{\mathcal{H}}$. The proper (C, h)-subdifferential of f at x_0 is given by

$$\partial^{\mathrm{PE}}_{C,h} f(x_0) := \{T \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n) : x_0 \in \mathrm{QPE}(f - T, \operatorname{dom} f, C, h)\}.$$

Each element $T \in \partial^{j}_{C,h} f(x_0)$ will be called a $(C, h)^j$-subgradient of f at x_0, for $j \in \{\mathrm{E}, \mathrm{WE}, \mathrm{PE}\}$.

Remark 12 The following inclusions are straightforward:

(a) $\partial^{\mathrm{E}}_{C,h} f(x_0) \subset \partial^{\mathrm{WE}}_{C,h} f(x_0)$ if $C \in \mathcal{H}$.
(b) $\partial^{\mathrm{PE}}_{C,h} f(x_0) \subset \partial^{\mathrm{E}}_{C+D\setminus\{0\},h} f(x_0)$ if $C \in \overline{\mathcal{H}}$.

Remark 13 Let us consider some particular cases.
(a) The scalar case. When $n = 1$ the three subdifferentials introduced in Definition 5 coincide. We deduce

$$(x^*, c) \in \partial^w f(x_0) \Leftrightarrow x^* \in \partial^{\mathrm{E}}_{(c,+\infty),h} f(x_0) = \partial^{\mathrm{WE}}_{(c,+\infty),h} f(x_0) = \partial^{\mathrm{PE}}_{(c,+\infty),h} f(x_0),$$

where $h(x, z) = \|x - z\|$ for $x, z \in \mathbb{R}^m$. It is also clear that when $D = \mathbb{R}_+$ and $C = (0, +\infty)$, one has

$$\partial^{\mathrm{E}}_{(0,+\infty),h} f(x_0) = \partial^{\mathrm{WE}}_{(0,+\infty),h} f(x_0) = \partial^{\mathrm{PE}}_{(0,+\infty),h} f(x_0) = \partial f(x_0),$$

while when $C = (\varepsilon, +\infty)$ with some $\varepsilon > 0$, one has

$$\begin{aligned} &\partial^{\mathrm{E}}_{(\varepsilon,+\infty),h} f(x_0) = \partial^{\mathrm{WE}}_{(\varepsilon,+\infty),h} f(x_0) = \partial^{\mathrm{PE}}_{(\varepsilon,+\infty),h} f(x_0), \\ &\partial^{\mathrm{E}}_{(\varepsilon,+\infty),1} f(x_0) = \partial^{\mathrm{WE}}_{(\varepsilon,+\infty),1} f(x_0) = \partial^{\mathrm{PE}}_{(\varepsilon,+\infty),1} f(x_0) = \partial_\varepsilon f(x_0), \end{aligned} \tag{22}$$

($h(x, z) = 1$, for all $x, z \in \mathbb{R}^m$, in statement (22)).

(b) Fréchet ε-subdifferential. Let $f:\mathbb{R}^m\to\mathbb{R}$. The Fréchet ε-subdifferential, denoted by $\partial^F_\varepsilon f$ and introduced by Kruger and Mordukhovich in [29] is given as follows: $x^*\in\partial^F_\varepsilon f(x_0)$ if and only if for each $\eta>0$, there exists $\delta>0$ such that

$$f(x)\geq f(x_0)+\langle x^*,x-x_0\rangle-(\varepsilon+\eta)\|x-x_0\|,\ \forall x\in \mathrm{B}(x_0,\delta), \tag{23}$$

where $\mathrm{B}(x_0,\delta)$ denotes the closed ball with center at x_0 and radius δ. It has a reach calculus and is very useful in obtaining approximate solutions of optimization problems (see, for instance, [26,35,39]). We prove that if $h(x,z)=\|x-z\|$, for all $x,z\in\mathbb{R}^m$ and $\varepsilon>0$, then

$$\partial^{\mathrm{E}}_{(\varepsilon,+\infty),h}f(x_0)\subset\partial^F_\varepsilon f(x_0).$$

Indeed, let $x^*\in\partial^{\mathrm{E}}_{(\varepsilon,+\infty),h}f(x_0)$. Then, by definition we have that

$$f(x)\geq f(x_0)+\langle x^*,x-x_0\rangle-\varepsilon\|x-x_0\|,\ \forall x\in\mathbb{R}^m.$$

Fix $\eta>0$. Then, in particular (23) holds for all $\delta>0$, and so $x^*\in\partial^F_\varepsilon f(x_0)$.

(c) El Maghri's subdifferential. If $C=q+D$, for $q\notin -D$, and $h(x,z)=1$, for all $x,z\in\mathbb{R}^m$, the efficient, weak and proper (C,h)-subdifferentials reduce, respectively, to the corresponding approximate subdifferentials introduced by El Maghri in [36].

(d) The case h is constant. If $h(x,z)=\varepsilon>0$ for all $x,z\in\mathbb{R}^m$, then $\partial^{\mathrm{PE}}_{C,h}f(x_0)\subset\partial^{\mathrm{Be}}_{C,\varepsilon}f(x_0)$, where $\partial^{\mathrm{Be}}_{C,\varepsilon}f(x_0)$ denotes the proper (C,ε)-subdifferential for vector mappings introduced in [13] (see also [14]), for which Moreau-Rockafellar type theorems were derived.
If in addition $C=\tilde{C}+\operatorname{int}K$, where $K\subset\mathbb{R}^n$ is a proper and solid cone such that $\tilde{C}\cap(-\operatorname{int}K)=\emptyset$, then the (C,h)-subdifferential (equal to the weak (C,h)-subdifferential) reduces to the weak $(\tilde{C},\varepsilon)$-subdifferential with respect to K introduced in [15, Definition 4.13]. It is used to characterize approximate proper solutions of a vector optimization problem whose objective mapping is given by a difference of two mappings, including, in particular, DC problems.

Remark 14 Assume $C\in\mathcal{H}$ is coradiant.
(a) Under the assumptions of Theorem 1, it is easy to see that

$$\partial^{\mathrm{E}}_{C,h}f(x_0)=\bigcap_{\varepsilon>0}\{T\in\mathcal{L}(\mathbb{R}^m,\mathbb{R}^n):x_0\in\mathrm{AE}(f-T,N_h(x_0,\varepsilon),C,\varepsilon)\}$$

(analogously for $\partial^{\mathrm{WE}}_{C,h}f(x_0)$, by replacing AE by WAE). For instance, if $\epsilon>0$, $f:\mathbb{R}^m\to\mathbb{R}$ and $h(x,z)=\|x-z\|$, for all $x,z\in\mathbb{R}^m$, we have that $x^*\in\partial^{\mathrm{E}}_{(\epsilon,+\infty),h}f(x_0)$ if and only if for every $\varepsilon>0$,

$$f(x)\geq f(x_0)+\langle x^*,x-x_0\rangle-\epsilon\varepsilon,\ \forall x\in\mathrm{B}(x_0,\varepsilon),$$

which points out the local behaviour of these subdifferentials. Analogously,

$$\partial^{\mathrm{PE}}_{C,h} f(x_0) = \bigcup_{D' \in \mathcal{G}(C)} \bigcap_{\varepsilon > 0} \{T \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n) : x_0 \in \mathrm{WAE}(f - T, N_h(x_0, \varepsilon), C + \mathrm{int}\, D', \varepsilon)\}.$$

(b) On the other hand, under the assumptions of Theorem 2 we deduce that

$$\bigcap_{n \in \mathbb{N}} \partial^{\mathrm{E}}_{C,h_n} f(x_0) = \{T \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n) : x_0 \in \mathrm{E}(f - T, \mathrm{dom}\, f, D)\} =: \partial^{\mathrm{E}} f(x_0)$$

(we obtain analogous equality for the weak case, by replacing E by WE). Also,

$$\bigcup_{D' \in \mathcal{G}(C)} \bigcap_{n \in \mathbb{N}} \partial^{\mathrm{PE}}_{C + \mathrm{int}\, D', h_n} f(x_0) = \{T \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n) : x_0 \in \mathrm{He}(f - T, \mathrm{dom}\, f, D)\} =: \partial^{\mathrm{He}} f(x_0).$$

Hence, one can reach exact subdifferentials in terms of the intersection of approximate subdifferentials for which the error measured by h_n converges to zero. Subdifferentials $\partial^{\mathrm{E}} f(x_0)$, $\partial^{\mathrm{WE}} f(x_0)$ and $\partial^{\mathrm{He}} f(x_0)$ were introduced by El Maghri and Laghdir in [37] (see also [36]).

In what follows, we suppose that $h(x, x) = 0$, for all $x \in \mathbb{R}^m$. Let us now use the method of linear scalarization to express the generalized subgradients of a vector mapping in terms of subgradients of the associated scalar mappings. Given $\xi \in \mathbb{R}^n \backslash \{0\}$ and $x_0 \in \mathrm{dom}\, f$, we define $l_{\xi, x_0} : \mathbb{R}^m \to \overline{\mathbb{R}}$ as

$$l_{\xi,x_0}(x) := (\xi \circ f)(x) + \kappa_C(\xi) h(x, x_0), \ \forall x \in \mathbb{R}^m.$$

Here are some relationships between the subdifferential of the function $l_{\xi,x_0}(x)$ and that of f.

Corollary 6 *Let $x_0 \in \mathrm{dom}\, f$ and $C \in \mathcal{H}$ and $f_T := f - T$, for $T \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$. The following statements hold.*

(a) $\bigcup_{\xi \in C^+ \backslash \{0\}} \{T \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n) : \xi \circ T \in \partial(l_{\xi,x_0})(x_0)\} \subset \partial^{\mathrm{WE}}_{C,h} f(x_0)$ *if* $\mathrm{int}\, C \neq \emptyset$. *Equality holds if C is coradiant and if either D is solid, $C + D = C$ and the mapping $x \to f_T(x) - f_T(x_0) + h(x, x_0)C$ is nearly D-subconvexlike on* $\mathrm{dom}\, f$ *for all $T \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$, or C is convex and* $\mathrm{cone}\, C$ *is solid, and the mapping $x \to f_T(x) - f_T(x_0) + h(x, x_0)C$ is nearly* $\mathrm{cone}\, C$*-subconvexlike on* $\mathrm{dom}\, f$ *for all $T \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$.*

(b) $\bigcup_{\xi \in D^{s+} \cap C^+} \{T \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n) : \xi \circ T \in \partial(l_{\xi,x_0})(x_0)\} \subset \partial^{\mathrm{PE}}_{C,h} f(x_0)$ *if $C \in \overline{\mathcal{H}}$. Equality holds if in addition the mapping $x \to f_T(x) - f_T(x_0) + h(x, x_0)C$ is nearly D-subconvexlike on* $\mathrm{dom}\, f$*, for all $T \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$.*

Proof The proof is straightforward from Definition 5, Theorem 4, and Corollaries 1 and 2.

By a similar way, the generalized proper subgradients can also be expressed as follows, by means of associated nonlinear scalarized functions.

Corollary 7 *Let $x_0 \in \operatorname{dom} f$.*

(a) If $q \in D\backslash\{0\}$, $\emptyset \neq E \subset \mathbb{R}^n\backslash\{0\}$, $D' \in \mathcal{G}(D\backslash\{0\})$ such that $E \cap (-D') = \emptyset$ and $C := E + D'$, then

$$\partial^{\mathrm{PE}}_{C,h} f(x_0) \subset \{T \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n) : x_0 \in \mathrm{SAE}_{\operatorname{dom} f}(f_T, \varphi_{D',q}, \kappa_E(\varphi_{D',q}))\}.$$

If additionally $E \subset \lambda q + D$ for some $\lambda > 0$, then

$$\partial^{\mathrm{PE}}_{C,h} f(x_0) \supset \{T \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n) : x_0 \in \mathrm{SAE}_{\operatorname{dom} f}(f_T, \varphi_{D',q}, \lambda)\}.$$

(b) If $q \in D\backslash\{0\}$ and $C = q + D$, then

$$\partial^{\mathrm{PE}}_{C,h} f(x_0) = \bigcup_{D' \in \mathcal{G}(D\backslash\{0\})} \{T \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n) : x_0 \in \mathrm{SAE}_{\operatorname{dom} f}(f_T, \varphi_{D',q}, 1)\}.$$

Proof The proof is a direct application of Theorem 7 and Corollary 4.

The next theorem provides an existence result of $(C,h)^{\mathrm{PE}}$-subgradients. The graph of the set valued mapping $x \mapsto f(x) + h(x, x_0)C$ is denoted $\mathrm{gr}(f + h(\cdot, x_0)C)$.

Theorem 8 *Suppose that $\operatorname{int}(D^+) \neq \emptyset$. Let $x_0 \in \mathbb{R}^m$ and let $C \in \overline{\mathcal{H}}$ be a coradiant set. If there exists a closed and convex cone $M \subset \mathbb{R}^m \times \mathbb{R}^n$ such that*

(a) $\mathrm{gr}(f + h(\cdot, x_0)C) - (x_0, f(x_0)) \subset M$,

(b) $M \cap (-(\{0\} \times D)) = \{(0,0)\}$,

then $\partial^{\mathrm{PE}}_{C,h} f(x_0) \neq \emptyset$,

Proof Since $\operatorname{int}(D^+) \neq \emptyset$ it follows that $\operatorname{int}((\{0\} \times D)^+) \neq \emptyset$. As (b) holds, it is clear that the hypotheses of [25, Theorem 3.22] are verified, so there exists $(x^*, y^*) \in \mathbb{R}^m \times \mathbb{R}^n\backslash\{(0,0)\}$ such that

$$-\langle y^*, d\rangle \leq \langle (x^*, y^*), t\rangle,\ \forall t \in M,\ \forall d \in D, \tag{24}$$

$$-\langle y^*, d\rangle < 0,\ \forall d \in D\backslash\{0\}. \tag{25}$$

From (25) we deduce that $y^* \in D^{s+}$. On the other hand, (24) implies that

$$\langle x^*, x - x_0\rangle + \langle y^*, f(x) + h(x, x_0)c - f(x_0)\rangle \geq 0,\ \forall x \in \operatorname{dom} f,\ \forall c \in C. \tag{26}$$

It follows that $y^* \in (\operatorname{cone} C)^+$. Otherwise, if there exists $\bar{c} \in C$ such that $\langle y^*, \bar{c}\rangle < 0$, then for a fixed $\bar{x} \neq x_0$ and for $\lambda >> 1$ we have that

$$\langle x^*, \bar{x} - x_0\rangle + \langle y^*, f(\bar{x}) + h(\bar{x}, x_0)(\lambda\bar{c}) - f(x_0)\rangle < 0,$$

which contradicts (26) for $x = \bar{x}$ and $c = \lambda\bar{c} \in C$ (C is coradiant). Thus, $y^* \in (\operatorname{cone} C)^+$.

Let $q \in D\backslash\{0\}$ such that $\langle y^*, q\rangle = 1$. We define $T(x) = -\langle x^*, x\rangle q \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$. Then, from (26) we deduce in particular that

$$(y^* \circ f)(x) + \kappa_C(y^*)h(x, x_0) \geq (y^* \circ f)(x_0) + (y^* \circ T)(x - x_0),\ \forall x \in \operatorname{dom} f,$$

i.e., $l_{y^*,x_0}(x) \geq l_{y^*,x_0}(x_0) + (y^* \circ T)(x - x_0)$, for all $x \in \operatorname{dom} f$, so $y^* \circ T \in \partial(l_{y^*,x_0})(x_0)$ and by Corollary 6(b) we conclude that $T \in \partial^{\mathrm{PE}}_{C,h} f(x_0)$. The proof is complete.

Example 4 Condition (a) of Theorem 8 is equivalent to

$$\{(x - x_0, f(x) - f(x_0) + h(x, x_0)c) : x \in \operatorname{dom} f, c \in C\} \subset M.$$

For instance, if $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x_1, x_2) = \sqrt{x_1^2 + x_2^2}$, for every $(x_1, x_2) \in \mathbb{R}^2$, $D = \mathbb{R}_+$, $C = (\varepsilon, +\infty)$, $\varepsilon > 0$, $x_0 = (0, 0)$ and $h(x, z) = \|x - z\|$, for $x, z \in \mathbb{R}^2$ ($\|\cdot\|$ denotes in this case the Euclidean norm), then this condition reduces to

$$\left\{\left(x_1, x_2, (c+1)\sqrt{x_1^2 + x_2^2}\right) : (x_1, x_2) \in \mathbb{R}^2, c \in (\varepsilon, +\infty)\right\} \subset M$$

and is satisfied for $M = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 \geq (\varepsilon + 1)\sqrt{x_1^2 + x_2^2}\right\}$, which is a closed and convex cone. Clearly, for this set M, condition (b) of Theorem 8 is verified. So by this result we know that $\partial^{\mathrm{PE}}_{C,h} f(0, 0) \neq \emptyset$. Indeed, following the definition, it is easy to see that $(0, 0) \in \partial^{\mathrm{PE}}_{C,h} f(0, 0)$.

Remark 15 It is clear from the definition that when $S = \operatorname{dom} f$, a necessary and sufficient condition for a point $x_0 \in S$ to be a (C, h)-quasi efficient (respectively weak efficient, proper efficient) solution is that the corresponding (C, h)-subdifferential at x_0 contains 0. This, however, is not the case when S is a proper subset of $\operatorname{dom} f$. More precisely, if $S \subset \operatorname{dom} f$ and $S \neq \operatorname{dom} f$, it is not necessary that (C, h)-subdifferential of f at a (C, h) quasi efficient solution contains 0 as it is shown in Example 5.

We give now a sufficient condition for (C, h)-quasi efficient solutions when S is a proper subset of $\operatorname{dom} f$.

Corollary 8 *Let $x_0 \in S \subset \operatorname{dom} f$. The following statements hold.*

(i) If $C \in \mathcal{H}$ and if $\partial^{\mathrm{E}}_{C,h} f(x_0)$ contains some $T \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$ such that

$$C + T(\operatorname{cone}(S - x_0)) \subset C, \tag{27}$$

then x_0 is a (C, h)-quasi efficient solution of $(\mathcal{P})$.

(ii) If $C \in \mathcal{H}$ is solid and if $\partial^{\mathrm{WE}}_{C,h} f(x_0)$ contains some $T \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$ such that

$$\operatorname{int} C + T(\operatorname{cone}(S - x_0)) \subset \operatorname{int} C, \tag{28}$$

then x_0 is a (C, h)-quasi weak efficient solution of $(\mathcal{P})$.

(iii) If $C \in \overline{\mathcal{H}}$ and if $\partial^{\mathrm{PE}}_{C,h} f(x_0)$ contains some $T \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$ such that (27) holds, then x_0 is a (C, h)-quasi proper efficient solution of $(\mathcal{P})$.

Proof For the first statement, suppose to the contrary that x_0 is not a (C, h)-quasi efficient solution. Then, there exist some $x \in S \backslash \{x_0\}$ and $c \in C$ such that $0 = f(x) - f(x_0) + h(x, x_0)c$. For $T \in \partial^{\mathrm{E}}_{C,h} f(x_0)$ satisfying (27), one has

$$h(x, x_0)c + T(x - x_0) \in h(x, x_0)(c + T(\operatorname{cone}(S - x_0))) \subset h(x, x_0)C.$$

It follows that

$$\begin{aligned} 0 &= f(x) - T(x) - (f(x_0) - T(x_0)) + h(x, x_0)c + T(x - x_0) \\ &\in (f - T)(x) - (f - T)(x_0) + h(x, x_0)C, \end{aligned}$$

which is a contradiction because $T \in \partial^{\mathrm{E}}_{C,h} f(x_0)$.
The second statement is proven by the same argument as for the first statement with $\operatorname{int} C$ replacing C.
To prove the third statement, suppose to the contrary that x_0 is not a (C,h)-quasi proper efficient solution of $(\mathcal{P})$. For every $D' \in \mathcal{G}(C)$, there exists some $\bar{x} \in S \backslash \{x_0\}$ such that

$$f(x_0) \in f(\bar{x}) + h(\bar{x}, x_0)(C + \operatorname{int} D'). \tag{29}$$

Let T be as given in (iii). By definition, x_0 is a (C,h)-quasi proper efficient solution for $f - T$ on $\operatorname{dom} f$, that is, there is some $D'' \in \mathcal{G}(C)$ such that no $x \in \operatorname{dom} f \setminus \{x_0\}$ satisfies

$$f(x_0) - T(x_0) \in f(x) - T(x) + h(x, x_0)(C + \operatorname{int} D''). \tag{30}$$

For $D' = D''$ we deduce from (29) and (27) that

$$\begin{aligned} f(x_0) - T(x_0) &\in f(\bar{x}) - T(\bar{x}) + T(\bar{x} - x_0) + h(\bar{x}, x_0)(C + \operatorname{int} D'') \\ &\subset f(\bar{x}) - T(\bar{x}) + h(\bar{x}, x_0)(C + T(\operatorname{cone}(S - x_0)) + \operatorname{int} D'') \\ &\subset f(\bar{x}) - T(\bar{x}) + h(\bar{x}, x_0)(C + \operatorname{int} D''), \end{aligned}$$

which contradicts (30). The proof is complete.

A particular case of Corollary 8 when $C = q + D$ with $q \notin -D$ and $h(x,z) = 1$ gives sufficient conditions for q-efficient solutions in terms of El Maghri's q-subdifferential [36,37]. Condition (27) takes a simple form:

$$T(S - x_0) \subset D. \tag{31}$$

In fact, (31) implies $C + T(cone(S - x_0)) = q + D + T(cone(S - x_0) \subset q + D + D \subset q + D = C$ which is (27). Conversely, if for some $x \in S$, $T(x - x_0) \notin D$, then $D + T(cone(S - x_0) \not\subset D$. Hence $C + T(cone(S - x_0) \not\subset C$.

We close this section with an example to show that the conditions given in Corollary 8 are sufficient, but not necessary for quasi efficient solutions.

Example 5 Consider $(\mathcal{P})$ in which $D = \mathbb{R}^2_+$, $C = D \setminus \{0\}$, $h(x,z) = |x - z|$ for $x, z \in \mathbb{R}$, $S = [-1, 1]$ and $f : \mathbb{R} \to \mathbb{R}^2$ defined by

$$f(x) = \begin{cases} (x, x^3) & \text{if } x \geq 0 \\ (x, -x) & \text{if } x \in [-1, 0) \\ (x, x) & \text{if } x < -1. \end{cases}$$

We claim that $x_0 = 0$ is a (C,h)-quasi efficient solution of $(\mathcal{P})$. Indeed, the inclusion $f(x_0) \in f(x) + h(x, x_0)C$ on $S \setminus \{0\}$ takes an explicit form as follows

$$(0,0) \in \begin{cases} (1, x^2) + C & \text{if } x \in (0, 1] \\ (-1, 1) + C & \text{if } x \in [-1, 0). \end{cases}$$

It is clear that this inclusion does not have solution. Hence x_0 is a (C,h)-quasi efficient solution of $(\mathcal{P})$. Now, let $T(x) = (a,b)x$ for every $x \in \mathbb{R}$ and some $(a,b) \in \mathbb{R}^2$.

If $(a,b) \neq (0,0)$, then $T(\mathrm{cone}(S - x_0))$ is a straight line in $\mathbb{R}^2$, and so (27) cannot hold because $C \subset \mathbb{R}^2_+$. If $(a,b) = (0,0)$, then $T \notin \partial^{\mathrm{E}}_{C,h} f(x_0)$, since the inclusion $f(x_0) - T(x_0) \in f(x) - T(x) + h(x, x_0)C$ has solution $x \neq 0$, for example, $x = -2$. Thus, there is no $T \in \partial^{\mathrm{E}}_{C,h} f(x_0)$ satisfying (27) and so the conditions given in Corollary 8 are not necessary for quasi efficient solutions.

6 Conclusions

We have defined general concepts of quasi efficiency that unify the most known notions of efficiency in multiobjective optimization. We have studied their properties and we have provided characterizations for these solutions through linear and nonlinear scalarization. The particular case when the ordering cone is polyhedral is also studied, for which the characterizations through scalarization are more suitable computationally.

By means of these notions, we have defined respective subdifferentials for vector mappings that reduce to well-known approximate and weak subdifferentials given in the literature. We have studied some basic properties and we have obtained necessary and sufficient conditions to determine their subgradients through scalarization. Finally, we have also provided a sufficient condition for the existence of subgradients of one of these subdifferentials. Further applications of new subdifferentials in vector optimization and in set optimization will be addressed in a future research.

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