Noname manuscript No.
(will be inserted by the editor)

Variants of the Ekeland Variational Principle for Approximate Proper Solutions of Vector Equilibrium Problems

L. P. Hai · L. Huerga · P. Q. Khanh · V. Novo

Received: date / Accepted: date

Abstract In this paper, we provide variants of the Ekeland variational principle for a type of approximate proper solutions of a vector equilibrium problem, whose final space is finite dimensional and partially ordered by a polyhedral cone. Depending on the choice of an approximation set that defines these solutions, we prove that they approximate suitably exact weak efficient/proper efficient/efficient solutions of the problem. The variants of the Ekeland variational principle are obtained for an unconstrained and also for a cone-constrained vector equilibrium problem, through a nonlinear scalarization, and expressed by means of the matrix that defines the ordering cone, which makes them easier to handle. At the end, the results are applied to multiobjective optimization problems, for which a related vector variational inequality problem is defined.

Keywords Ekeland variational principle · vector equilibrium problems · approximate proper solutions · multiobjective optimization · variational inequalities

Mathematics Subject Classification (2010) 90C33 · 90C26 · 90C29 · 49J52 · 49J53

L. P. Hai
Department of Mathematics and Computing, University of Science, Vietnam National University-Hochiminh City, Hochiminh City, Vietnam
E-mail: lephuochai88@gmail.com

L. Huerga, V. Novo
Departamento de Matemática Aplicada, E.T.S.I. Industriales, Universidad Nacional de Educación a Distancia, c/ Juan del Rosal 12, Ciudad Universitaria, 28040 Madrid, Spain
E-mail: lhuerga@ind.uned.es, vnovo@ind.uned.es

P. Q. Khanh
Department of Mathematics, International University, Vietnam National University-Hochiminh City, Linh Trung, Thu Duc, Hochiminh City, Vietnam
E-mail: pqkhanh@hcmiu.edu.vn

1 Introduction

The Ekeland variational principle (briefly, EVP) [9] is a cornerstone and one of the most important results of optimization and variational analysis and many other areas in the last decades. There have been many variants extending this seminal principle to various contexts, see for instance few references from different development periods, [3,25,27,33,37,39,40]. The EVP has an amazingly countless of applications in an impressively wide range of areas, see, for instance, [8,30,31,35,46,50]. In particular, we mention the following contributions, which are closely related to the topic of this paper. The first work on the EVP for bifunctions is [42]. Such results are in fact mathematically equivalent to variants with unifunctions i.e., functions of one component, see [2], but they are significant for various applications, see, e.g., [5,26, 44]. In this paper, we develop variants of the EVP for vector equilibrium problems and a type of approximate proper efficiency notion.

Regarding notions of solutions, for vector equilibrium problems, the basic solution concept is the efficient solution. But sets of efficient solutions are usually too big, including anomalous ones. This fact makes necessary to define a more restrictive notion of efficiency. Due to this, the concepts of proper efficiency arise in the literature (see, for instance, [4,11,14,15,29,34]). These notions provide a selected collection of solutions that satisfy better properties. For instance, in [14], Gong introduced a type of proper efficient solutions in the sense of Henig (see [29]) for vector equilibrium problems that can be characterized through a linear scalarization, under generalized convexity assumptions, and also through a nonlinear scalarization with no convexity hypotheses (see [16]).

However, in practice, mathematical models only approximate practical problems and so their exact solutions, which are often difficult to be computed, may be not more useful than approximate ones. Thus, approximate efficiency is natural and inevitable in applied mathematics.

Due to the above, we are interested in working with an appropriate notion of approximate proper efficiency for vector equilibrium problems, that provides a collection of approximate solutions which represents suitably the set of exact solutions. This notion, which was introduced by Ródenas-Pedregosa in [47], is based on the concept of approximate proper efficiency in the sense of Henig given in [21] for vector optimization problems, and is defined by means of a nonempty set which represents an approximation of the ordering cone, and a nonnegative scalar, that measures the size of the error.

In this paper, we will see that for specific approximation sets of the ordering cone, a sequence of approximate proper solutions tends to an exact efficient/weak efficient/proper efficient solution of the vector equilibrium problem, when the error goes to zero. Thus, depending on the choice of the approximation set, we can reach different types of exact solutions.

The study of the limit behaviour of approximate solutions when the error tends to zero is relevant. For instance, there are simple problems for which the approximate efficiency notion introduced by Kutateladze [38], which is probably the most known, provides an unbounded set of approximate solutions, while the exact efficient set is bounded (see, for instance [19, Example 3.10]). A more complete study of the limit behaviour of approximate solutions can be found in [23] for the particular case of vector optimization problems.

In this work, we deal with two variants of the EVP for the aforementioned type of approximate proper solutions of a vector equilibrium problem, whose associated bifunction has finite dimensional images. For this aim, we consider that the ordering cone is polyhedral, which lets us express the approximate proper solutions in terms of a family of dilating cones introduced by Kaliszewski [32], that are defined by means of perturbations of the matrix that defines the ordering cone.

We study the case of an unconstrained vector equilibrium problem, and the case when the decision set is given by a cone constraint. In both situations, the variants of the EVP are obtained through a nonlinear scalarization. This fact makes the results presented in this paper interesting, not only because we convert a vector problem to a scalar one, which is easier to handle, but also because they are expressed in terms of the matrix that defines the ordering cone, which is interesting from a computational point of view.

Furthermore, as an application, we define an approximate vector variational inequality problem, defined by means of an approximate strong subdifferential introduced in [20]. This problem extends to the approximate case well-known exact vector variational inequality problems related to vector optimization problems (see, for instance, [7,51]).

The paper is structured as follows. In Section 2 we state the framework, notations and definitions that we need along the paper. In Section 3 we analyze the limit behaviour of the approximate proper solutions for vector equilibrium problems. In Section 4, we provide a variant of the EVP for unconstrained vector equilibrium problems, while in Section 5 we obtain the corresponding result for cone-constrained vector equilibrium problems. As an application, in Section 6, we derive an Ekeland variational principle for a multiobjective optimization problem and we define and study a new approximate vector variational inequality problem, from which we obtain a sufficient condition for approximate proper solutions of the multiobjective optimization problem. Finally, in Section 7 we state some conclusions.

2 Preliminaries

In this paper, we use the standard notation. Indeed, $\mathbb{N}$ is the set of the positive integers and $\mathbb{R}^n_+$ is the nonnegative orthant of an n-dimensional space $\mathbb{R}^n$. For a subset E on a topological vector space, $\operatorname{int} E$, $\operatorname{cl} E$, $\operatorname{bd} E$, $\operatorname{co} E$ and $\operatorname{cone} E$ stand for the interior, the closure, the boundary, the convex hull of E, and the cone generated by E, respectively. Also, it is said that E is coradiant if $E = \bigcup_{\alpha \geq 1} \alpha E$.

In what follows if not specified differently, we consider a nontrivial complete metric space (X,d) and a vector bifunction (also called a bimap) $F : X \times X \to \mathbb{R}^n$. In $\mathbb{R}^n$, we establish an order given by an ordering cone $\{0\} \neq D \neq \mathbb{R}^n$ in the usual way, i.e.,

$$y_1, y_2 \in \mathbb{R}^n,\ y_1 \leq_D y_2 \Longleftrightarrow y_2 - y_1 \in D.$$

Along the work, we suppose that D is polyhedral, that is

$$D := \{y \in \mathbb{R}^n : Ay^t \in \mathbb{R}^p_+\},$$

where y^t denotes the transpose of y, $A \in \mathscr{M}_{p\times n}$ (i.e., A has p rows and n columns), $p \geq n$ and $\operatorname{rank}(A) = n$. This last condition is equivalent to the pointedness of D. Thus, D is convex, closed and pointed, so it induces a partial order in $\mathbb{R}^n$.

The polar cone and the strict polar cone of D are denoted, respectively, by D^+ and D^{s+}, i.e.,

$$D^+ := \{\mu \in \mathbb{R}^n : \langle \mu, d\rangle \geq 0,\ \forall d \in D\},$$
$$D^{s+} := \{\mu \in \mathbb{R}^n : \langle \mu, d\rangle > 0,\ \forall d \in D\backslash\{0\}\}.$$

In this paper, we focus on the classical vector equilibrium problem (briefly, VEP) defined as follows

$$\text{Find } x_0 \in X \text{ such that } F(x_0, X) \cap (-D\backslash\{0\}) = \emptyset. \qquad (\mathscr{VEP})$$

The efficient set, that is, the set of points $x_0 \in X$ that are solutions of $(\mathscr{VEP})$ is denoted by $\mathrm{E}(F, X, D)$.

If we suppose that D is solid, i.e., $\operatorname{int} D \neq \emptyset$, then it is said that $x_0 \in X$ is a weak efficient solution of $(\mathscr{VEP})$ if $x_0 \in \mathrm{E}(F, X, \operatorname{int} D \cup \{0\})$. The set of weak efficient solutions will be denoted by $\mathrm{WE}(F, X, D)$. In the literature for $(\mathscr{VEP})$, bimap F is said to satisfy the diagonal null property if $F(x, x) = 0$ for all $x \in X$.

The following notion of proper efficiency for $(\mathscr{VEP})$ was defined by Gong [14] and it is based on the concept of proper efficiency given by Henig [29] for vector optimization problems. Let

$$\mathscr{G} := \left\{D' \subsetneq \mathbb{R}^n : D' \text{ is a solid convex cone, } D\backslash\{0\} \subset \operatorname{int} D'\right\}.$$

Definition 1 It is said that $x_0 \in X$ is a Henig proper efficient solution of $(\mathscr{VEP})$, and it is denoted by $x_0 \in \mathrm{He}(F, X, D)$, if there exists $D' \in \mathscr{G}$ such that $x_0 \in \mathrm{WE}(F, X, D')$.

It is clear that $\mathrm{He}(F, X, D) \subset \mathrm{E}(F, X, D) \subset \mathrm{WE}(F, X, D)$, where for the last inclusion D is considered to be solid. In this paper, we deal with the next definition of approximate proper efficiency for vector equilibrium problems, which was introduced in [47], and is based on the notion of approximate proper efficiency in the sense of Henig defined in [21] for vector optimization problems.

First of all, for a nonempty set $C \subset \mathbb{R}^n\backslash\{0\}$, we define the set-valued mapping $C : \mathbb{R}_+ \rightrightarrows \mathbb{R}^n$ by

$$C(\varepsilon) := \begin{cases} \varepsilon C & \text{if } \varepsilon > 0, \\ (\operatorname{cone} C)\backslash\{0\} & \text{if } \varepsilon = 0, \end{cases}$$

and also the following sets

$$\overline{\mathscr{H}} := \{\emptyset \neq C \subset \mathbb{R}^n\backslash\{0\} : \operatorname{cl} \operatorname{cone} C \cap (-D) = \{0\}\},$$
$$\mathscr{G}(C) := \left\{D' \subset \mathbb{R}^n : D' \text{ is a solid convex cone, } D\backslash\{0\} \subset \operatorname{int} D', C \cap (-\operatorname{int} D') = \emptyset\right\}.$$

Definition 2 Let $\varepsilon \geq 0$ and $C \in \overline{\mathscr{H}}$. It is said that a point $x_0 \in X$ is a Henig (C, ε)-proper efficient solution of $(\mathscr{VEP})$, and it is denoted by $x_0 \in \mathrm{He}(F, X, C, \varepsilon)$, if there exists $D' \in \mathscr{G}(C)$ such that

$$F(x_0, X) \cap (-C(\varepsilon) - \operatorname{int} D') = \emptyset. \qquad (1)$$

Remark 1 (a) Conditions $C \in \overline{\mathscr{H}}$ and $D' \in \mathscr{G}(C)$ are required in order to give a consistent definition of approximate proper efficiency. Indeed, for instance, if F satisfies the diagonal null property, then statement (1) implies $\operatorname{cl}\operatorname{cone} C \cap (-\operatorname{int} D') = \emptyset$.

(b) Let $C' := C + \operatorname{int} D'$. Observe that $C'(\varepsilon) = C(\varepsilon) + \operatorname{int} D'$, for all $\varepsilon \geq 0$ and $\operatorname{He}(F,X,C,\varepsilon) = \operatorname{He}(F,X,C+D,\varepsilon)$. The set $C+D$ may be understood as an approximation of D, and C' as a dilating approximation of D. Moreover, if $C = D\backslash\{0\}$ in the above definition, then we recover the notion of exact Henig proper efficiency for $(\mathscr{V}\mathscr{E}\mathscr{P})$.

(c) The Henig (C,ε)-proper efficient solutions of $(\mathscr{V}\mathscr{E}\mathscr{P})$ have been characterized through linear scalarization, under generalized convexity hypotheses (see [47, Theorems 2.18 and 2.19]).

(d) Let $f : X \to \mathbb{R}^n$ and consider the vector optimization problem that consists in minimizing f on X, with respect to the order given by the cone D. This problem is equivalent to $(\mathscr{V}\mathscr{E}\mathscr{P})$ for $F(x,y) = f(y) - f(x)$. In this particular case, the concept of (C,ε)-proper efficiency for $(\mathscr{V}\mathscr{E}\mathscr{P})$ reduces to the notion of approximate proper efficiency due to Gutiérrez, Huerga, Jiménez and Novo [21] for the vector optimization problem.

In the following example we show the applicability of the Henig (C,ε)-proper solutions, for a suitable set C.

Example 1 Let $X = \mathbb{R}^2$, $Y = \mathbb{R}^2$, $D = \mathbb{R}^2_+$, $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the function defined as $f(x_1,x_2) = (x_1,x_2)$ if $(x_1,x_2) \in \mathbb{R}^2_+$, $f(x_1,x_2) = (1,1)$, otherwise; and let

$$F((x_1,x_2),(y_1,y_2)) = f(y_1,y_2) - f(x_1,x_2).$$

It is clear that $\operatorname{E}(F,X,D) = \{(0,0)\}$ and $\operatorname{WE}(F,X,D) = \operatorname{bd}\mathbb{R}^2_+$. Let us note that the weak efficient set is unbounded, whereas the efficient set is just formed by point $(0,0)$.

However, if we consider $C = \operatorname{co}\{(1,0),(0,1)\} + D$, and $0 < \varepsilon < 2$, then one can easily see that $\operatorname{He}(F,X,C,\varepsilon) = \{(x_1,x_2) \in \mathbb{R}^2_+ : x_2 \leq \varepsilon - x_1\}$. It is clear that this set of approximate proper solutions is bounded and represents a good approximation of the efficient set, for ε small enough. In the next section, we will study the limit behaviour of the approximate proper solutions when ε goes to zero, for specific sets C.

Given $\rho \geq 0$, the following dilating cone was introduced by Kaliszewski (see [32])

$$D_\rho := \{y \in \mathbb{R}^n : Ay^t + \rho u^t u A y^t \in \mathbb{R}^p_+\},$$

where u denotes the p-dimensional row vector $(1,1,\ldots,1)$. Note that D_ρ is convex, closed and pointed, and $D\backslash\{0\} \subset \operatorname{int} D_\rho$ for all $\rho > 0$. Also, observe that $D_0 = D$.

Let $\xi := uA$. It follows that $\xi \in D^{s+}_\rho$, for all $\rho \geq 0$ and

$$B_\rho := \{y \in D_\rho : \langle \xi, y \rangle = 1\}$$

is a compact base of D_ρ, for all $\rho \geq 0$. If $\rho = 0$, we denote $B := B_0$.

The following lemma was stated in [32, Lemma 3.7].

Lemma 1 *For any closed cone $D' \subset \mathbb{R}^n$ such that $D\backslash\{0\} \subset \operatorname{int} D'$, there exists $\rho > 0$ such that $D_\rho\backslash\{0\} \subset \operatorname{int} D'$.*

Remark 2 Let us note that by Lemma 1 we have

$$\mathrm{He}(F,X,C,\varepsilon) = \mathrm{He}(F,X,C+D,\varepsilon) = \bigcup_{\substack{\rho>0 \\ D_\rho \in \mathscr{G}(C)}} \mathrm{He}(F,X,C+D_\rho,\varepsilon),$$

$$\mathrm{He}(F,X,D) = \bigcup_{\rho>0} \mathrm{He}(F,X,D_\rho).$$

3 Limit behaviour of (C,ε)-proper efficient solutions of VEP

In this section, we study the limit behaviour of the (C,ε)-proper solutions of $(\mathscr{V}\mathscr{E}\mathscr{P})$ when ε tends to zero for some specific and fixed sets $C \in \overline{\mathscr{H}}$. For this aim, along the section, we suppose that for each $y \in X$ the map $f_y(x) := F(x,y)$ is continuous on X.

We have the following result, when $C = q_\rho + D_\rho$, $q_\rho \in D_\rho \backslash \{0\}$, and $\rho \geq 0$.

Theorem 1 *Let $\rho \geq 0$, $q_\rho \in D_\rho \backslash \{0\}$, $x_0 \in X$, and sequences $(x_k) \subset X$, $(\varepsilon_k) \subset \mathbb{R}_+ \backslash \{0\}$ such that $x_k \in \mathrm{He}(F,X,q_\rho + D_\rho,\varepsilon_k)$, for all $k \in \mathbb{N}$, $\varepsilon_k \to 0$ and $x_k \to x_0$.*

(a) *If $\rho > 0$, then $x_0 \in \mathrm{He}(F,X,D)$,*

(b) *If D is solid, then $x_0 \in \mathrm{WE}(F,X,D)$.*

Proof (a) For each $k \in \mathbb{N}$, since $x_k \in \mathrm{He}(F,X,q_\rho + D_\rho,\varepsilon_k)$, by definition and Remark 2 we deduce that there exists $\bar{\rho}_k > 0$ such that

$$F(x_k,x) \notin -\varepsilon_k q_\rho - D_\rho - \mathrm{int} D_{\bar{\rho}_k} = -\varepsilon_k q_\rho - \mathrm{int} D_{\max\{\rho,\bar{\rho}_k\}},\ \forall x \in X,$$

which implies

$$F(x_k,x) \notin -\varepsilon_k q_\rho - \mathrm{int} D_\rho,\ \forall x \in X. \tag{2}$$

Let $C := q_\rho + D_\rho$ and $\tilde{C} := \mathrm{int} C$. It is clear that $\tilde{C}$ is an open coradiant set and $\tilde{C}(0) = \mathrm{int} D_\rho$. It follows that $x_0 \in \mathrm{WE}(F,X,D_\rho)$. Indeed, reasoning by contradiction suppose that there exists $\bar{x} \in X$ such that $F(x_0,\bar{x}) \in -\mathrm{int} D_\rho$. Then, there exists $\alpha > 0$ such that $F(x_0,\bar{x}) \in -\tilde{C}(\alpha)$ and since $\tilde{C}(\alpha)$ is open and $F(\cdot,\bar{x})$ is continuous on X, there exists $k_0 \in \mathbb{N}$ such that

$$F(x_k,\bar{x}) \in -\tilde{C}(\alpha),\ \forall k \geq k_0. \tag{3}$$

Moreover, as $\varepsilon_k \to 0$, there exists $k_1 \in \mathbb{N}$ such that $\varepsilon_k \leq \alpha$, for all $k \geq k_1$, and since $\tilde{C}$ is coradiant, by [24, Lemma 3.1(ii)] we have that $\tilde{C}(\varepsilon_k) \supset \tilde{C}(\alpha)$, for all $k \geq k_1$, so by (3) we obtain that

$$F(x_k,\bar{x}) \in -\tilde{C}(\varepsilon_k),\ \forall k \geq \max\{k_0,k_1\},$$

which contradicts (2). Hence, $x_0 \in \mathrm{WE}(F,X,D_\rho)$, which implies that $x_0 \in \mathrm{He}(F,X,D)$.

(b) If $\rho > 0$, the assertion is trivial, since $\mathrm{He}(F,X,D) \subset \mathrm{WE}(F,X,D)$. For $\rho = 0$, it is clear that in particular $F(x_k,X) \cap (-q_0 - \mathrm{int} D) = \emptyset$. Let $C := q_0 + D$ and $\tilde{C} := \mathrm{int} C$. This set is open, coradiant and $\tilde{C}(0) = \mathrm{int} D$, so by reasoning in analogous way as in part (a) for this set $\tilde{C}$, we conclude that $x_0 \in \mathrm{WE}(F,X,D)$.

In the next result, we study the limit behaviour when $\varepsilon \to 0$ for the sets $C = B_\rho + D_{\bar{\rho}}$ with $\rho,\bar{\rho} \geq 0$. For this aim, we need the following lemma, which is based on [23, Lemma 3].

Lemma 2 *Consider two sequences $(\varepsilon_k) \subset \mathbb{R}_+ \backslash \{0\}$ and $(y_k) \subset \mathbb{R}^n$ be such that $\varepsilon_k \to 0$, $y_k \to y \in \mathbb{R}^n$, and*

$$y_k \in D \cap (\mathbb{R}^n \backslash (\varepsilon_k B + (D\backslash\{0\}))), \quad \forall k \in \mathbb{N}.$$

Then, $y = 0$.

Proof Under the given hypotheses, it follows that $\langle \xi, y_k \rangle \leq \varepsilon_k$, for all $k \in \mathbb{N}$. Indeed, if there exists $k \in \mathbb{N}$ such that $\langle \xi, y_k \rangle > \varepsilon_k$ then by considering $z_k := \frac{\varepsilon_k}{\langle \xi, y_k \rangle} y_k \in \varepsilon_k B$, we have that

$$y_k = z_k + \left(1 - \frac{\varepsilon_k}{\langle \xi, y_k \rangle}\right) y_k \in \varepsilon_k B + D\backslash\{0\},$$

which is a contradiction. Thus, $0 \leq \langle \xi, y_k \rangle \leq \varepsilon_k$, for all $k \in \mathbb{N}$, and since $\varepsilon_k \to 0$, we deduce that $\langle \xi, y_k \rangle \to \langle \xi, y \rangle = 0$. As D is closed, we have that $y \in D$, and since $\xi \in D^{s+}$, we finally conclude that $y = 0$.

Remark 3 Note that assumption $y_{k+1} \leq_D y_k$, for all k in [23, Lemma 3] is not required in Lemma 2.

In what follows for $(\mathscr{V}\mathscr{E}\mathscr{P})$, we usually impose one of the following hypotheses on the relation between the two components of F. For any $x, y, z \in X$,

$(\mathscr{A}_D)$ If $F(x,z) \in -D$ and $F(z,y) \in -D$, and $(F(x,z), F(y,z)) \neq (0,0)$, then

$$F(x,y) \leq_D F(x,z) + F(z,y);$$

$(\mathscr{B}_D)$ If $F(x,z) \in -D$ and $F(z,y) \in -D$, and $(F(x,z), F(y,z)) \neq (0,0)$, then

$$F(x,y) \in -D.$$

Note that in the studies of problems involving a bimap F in general and of the EVP for bifunctions/bimaps in particular, the following triangle inequality property is usually assumed:

$$F(x,y) \leq_D F(x,z) + F(z,y), \ \forall x, y, z \in X.$$

The implications [the triangle inequality property] $\Rightarrow$ $(\mathscr{A}_D) \Rightarrow (\mathscr{B}_D)$ are evident. The following simple examples show that the reverse implications do not hold.

Example 2 Let $D = \mathbb{R}^2_+$ and $F : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$F(x,y) = \begin{cases} (1,1) & \text{if } x > y, \\ (x-y, 0) & \text{if } x \leq y. \end{cases}$$

Then, F satisfies the diagonal null property. Moreover, $F(x,z) \in -D$, $F(z,y) \in -D$, and not both $(0,0)$ imply that $F(x,z) = (x-z, 0)$, $F(z,y) = (z-y, 0)$ and $x \leq z \leq y$. Hence, hypothesis $(\mathscr{A}_D)$ is satisfied as

$$F(x,z) + F(z,y) - F(x,y) = (x-z,0) + (z-y,0) - (x-y,0) = (0,0) \in \mathbb{R}^2_+.$$

On the other hand, the triangle inequality property is violated because, for $(x,y,z) = (2,1,5)$, one has

$$F(x,z) + F(z,y) - F(x,y) = (-3,0) + (1,1) - (1,1) = (-3,0) \notin \mathbb{R}^2_+.$$

Example 3 Let $D = \mathbb{R}^2_+$ and $F : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$F(x,y) = \begin{cases} (1,1) & \text{if } x > y, \\ (-1,-1) & \text{if } x \leq y. \end{cases}$$

It is not difficult to check that $(\mathscr{B}_D)$ is fulfilled, but $(\mathscr{A}_D)$ is not.

Theorem 2 *Let $\rho, \bar{\rho} \geq 0$, $x_0 \in X$, and sequences $(x_k) \subset X$, $(\varepsilon_k) \subset \mathbb{R}_+ \backslash \{0\}$ such that $x_k \in \mathrm{He}(F,X,B_\rho + D_{\bar{\rho}}, \varepsilon_k)$ for all $k \in \mathbb{N}$, $\varepsilon_k \to 0$, and $x_k \to x_0$.*

(a) *If $\bar{\rho} > 0$, then $x_0 \in \mathrm{He}(F,X,D)$,*

(b) *If F satisfies the hypothesis $(\mathscr{B}_D)$ and $F(x_k,x_0) \leq_D 0$ for all $k \in \mathbb{N}$, then $x_0 \in \mathrm{E}(F,X,D)$.*

(c) *If D is solid, then $x_0 \in \mathrm{WE}(F,X,D)$.*

Proof Let $q \in B$. Since $q \in B \subset B_{\hat{\rho}}$ for all $\hat{\rho} \geq 0$, it follows that

$$\mathrm{He}(F,X,B_\rho + D_{\bar{\rho}}, \varepsilon_k) \subset \mathrm{He}(F,X,q + D_{\bar{\rho}}, \varepsilon_k),\ \forall k \in \mathbb{N}.$$

Thus, part (a) follows by Theorem 1(a) and, analogously, part (c) follows by Theorem 1(b).

To prove part (b), observe first that, from part (a), if $\bar{\rho} > 0$ then $x_0 \in \mathrm{He}(F,X,D) \subset \mathrm{E}(F,X,D)$. So we just need to prove the case when $\bar{\rho} = 0$. As $x_k \in \mathrm{He}(F,X,B_\rho + D, \varepsilon_k)$ for all $k \in \mathbb{N}$, we have in particular that

$$F(x_k,X) \cap (-\varepsilon_k B_\rho - D\backslash\{0\}) = \emptyset,\ \forall k \in \mathbb{N}. \tag{4}$$

Suppose by reasoning to the contrary that there exists $\bar{x} \in X$ such that $F(x_0,\bar{x}) \in -D\backslash\{0\}$. By (4) we have

$$F(x_k,\bar{x}) \in \mathbb{R}^n \backslash (-\varepsilon_k B_\rho - D\backslash\{0\}) \subset \mathbb{R}^n \backslash (-\varepsilon_k B - D\backslash\{0\}). \tag{5}$$

Moreover, since F satisfies hypothesis $(\mathscr{B}_D)$,

$$F(x_k,\bar{x}) \in -D.$$

Thus, taking into account this and statement (5), we deduce by Lemma 2 that

$$\lim_{k\to\infty} F(x_k,\bar{x}) = F(x_0,\bar{x}) = 0,$$

which is a contradiction. The proof is complete.

Remark 4 (i) Note that in this section, it is not necessary to suppose that (X,d) is complete.

(ii) Theorems 1 and 2 extend [23, Corollaries 4 and 5] from a vector optimization problem to a vector equilibrium problem.

4 An EVP for unconstrained vector equilibrium problems

In this section, we present a variant of the EVP for (C,ε)-proper efficient solutions of $(\mathscr{VEP})$. The result is obtained through scalarization by using the version of the classical scalar EVP (see [9]) given in Lemma 3 below. Here, the classical lower semicontinuity of a function $\psi : X \to \mathbb{R} \cup \{\infty\}$ is replaced by the strict-decreasing lower semicontinuity defined as follows: ψ is called strictly-decreasingly lower semicontinuous if for $x \in X$ and $x_m \to x$ such that $\psi(x_{m+1}) < \psi(x_m)$ for all m, then $\psi(x) \leq \psi(x_m)$ for all m.

Lemma 3 (Theorem 6, [1]) *Let $\varepsilon > 0$, $x \in X$, and $\psi : X \to \mathbb{R} \cup \{\infty\}$ be a proper, strictly-decreasingly lower semicontinuous and bounded from below mapping such that*

$$\psi(x) \leq \inf_X \psi + \varepsilon.$$

Then, for any $\lambda > 0$, there exists $z \in X$ such that

(i) $d(z,x) \leq \lambda$;
(ii) $\psi(z) \leq \psi(x) - \dfrac{\varepsilon}{\lambda} d(z,x)$;
(iii) $\psi(u) + \dfrac{\varepsilon}{\lambda} d(z,u) > \psi(z)$ *for all $u \neq z$.*

Let $\emptyset \neq H \subset \mathbb{R}^n$ and $y_0 \in \mathbb{R}^n \backslash \{0\}$. The nonlinear functional $\varphi_{H,y_0} : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ is defined by Gerstewitz/Tammer and Iwanov in [12] as

$$\varphi_{H,y_0}(y) := \inf\{t \in \mathbb{R} : y \in ty_0 - H\},\ \forall y \in \mathbb{R}^n,$$

where it is understood that $\varphi_{H,y_0}(y) = \infty$ if $\{t \in \mathbb{R} : y \in ty_0 - H\} = \emptyset$. This functional is usually called "the smallest strictly monotonic functional" and it has been frequently used, overall in scalarization techniques to solve non convex vector optimization problems (see, for instance, [48,41,12,43]).

In what follows, we denote by e_i the i-th canonical row vector of $\mathbb{R}^p$. In the next lemma, proved in [22, Lemma 2.3], we provide the explicit expression of φ_{D_ρ,y_0}.

Lemma 4 *Let $\rho > 0$ and $y_0 \in D\backslash\{0\}$. Then,*

$$\varphi_{D_\rho,y_0}(y) = \max_{i \in \{1,2,\ldots,p\}} \left\{ \frac{\alpha_i \cdot y}{\alpha_i \cdot y_0} \right\},\ \forall y \in \mathbb{R}^n, \tag{6}$$

where $\alpha_i := (\rho u + e_i)A$, for all $i \in \{1,2,\ldots,p\}$.

Remark 5 By formula (6) or by applying [18, Theorem 2.3.1] it is easy to see that φ_{D_ρ,y_0}, for $\rho > 0$ and $y_0 \in D\backslash\{0\}$, is finite-valued, convex, positively homogeneous, D_ρ-monotone (i.e., $y_2 - y_1 \in D_\rho \Longrightarrow \varphi_{D_\rho,y_0}(y_1) \leq \varphi_{D_\rho,y_0}(y_2)$) and subadditive in $\mathbb{R}^n$. Moreover, for all $r \in \mathbb{R}$,

$$\{y \in \mathbb{R}^n : \varphi_{D_\rho,y_0}(y) < r\} = ry_0 - \operatorname{int} D_\rho, \tag{7}$$

$$\{y \in \mathbb{R}^n : \varphi_{D_\rho,y_0}(y) = r\} = ry_0 - \operatorname{bd} D_\rho, \tag{8}$$

$$\varphi_{D_\rho,y_0}(y + ry_0) = \varphi_{D_\rho,y_0}(y) + r,\ \forall y \in \mathbb{R}^n. \tag{9}$$

Let $y_0 \in D\backslash\{0\}$. Given $\rho, r > 0$, we define

$$\mathscr{A}_{\rho,r} := \{C \in \overline{\mathscr{H}} : C \cap (ry_0 - \operatorname{int} D_\rho) = \emptyset\},$$
$$\mathscr{A} := \bigcup_{\rho,r>0} \mathscr{A}_{\rho,r}.$$

Remark 6 (a) By statement (7), we deduce that $C \in \mathscr{A}_{\rho,r}$ if and only if $\inf_{c\in C} \varphi_{D_\rho,y_0}(c) \geq r$.

(b) Let $C \in \overline{\mathscr{H}}$. If there exists $\rho > 0$ such that $0 \notin \operatorname{cl}(C + D_\rho)$, then $C \in \bigcup_{r>0} \mathscr{A}_{\rho,r}$. Indeed, if $C \notin \bigcup_{r>0} \mathscr{A}_{\rho,r}$, then there exist $(r_n) \subset \mathbb{R}_+\backslash\{0\}$, $r_n \to 0$, $(c_n) \subset C$ and $(d_n) \subset \operatorname{int} D_\rho$ such that $c_n + d_n = r_n y_0$ for all n. Thus, by taking the limit, we deduce that $c_n + d_n \to 0$, so $0 \in \operatorname{cl}(C + D_\rho)$, and we reach a contradiction.

(c) For instance, the set $C = H + D_{\bar\rho}$, where $\bar\rho \geq 0$ and $H \subset D_{\bar\rho}\backslash\{0\}$ is compact, belongs to $\mathscr{A}$. Indeed, $C + D_\rho$ is closed for all $\rho > 0$ and $0 \notin C + D_\rho$, so by part (b) we conclude that $C \in \mathscr{A}$.

To obtain our variants of the EVP for problem $(\mathscr{VEP})$, a lower semicontinuity hypothesis is needed. Now we propose the following concept of lower semicontinuity.

Definition 3 Let $\emptyset \neq H \subset \mathbb{R}^n$ be a closed convex cone, $b \in \mathbb{R}^n\backslash\{0\}$, and $f : X \to \mathbb{R}^n$. f is called (b,H)-quasi lower semicontinuous from above ((b,H)-qlsca) at $x \in X$ if for each $r \in \mathbb{R}$ and $x_m \to x$, from $f(x_m) + rb \leq_H 0$ and $f(x_m) \not\leq_H f(x_{m+1})$ (i.e., $f(x_m) - f(x_{m+1}) \notin -H$) for all $m \in \mathbb{N}$, it follows that $f(x) + rb \leq_H 0$.

From now on if a property is satisfied for all $x \in X$ or, furthermore for all $b \in \mathbb{R}^n \setminus \{0\}$, we omit "at x" or "b', respectively.

We recall several concepts of lower semicontinuity in the literature, which are close to Definition 3, and discuss relations between them.

Definition 4 Let X, H, b, and f be as in Definition 3.

(i) ([49]) f is called (b,H)-lower semicontinuous ((b,H)-lsc) if the set $M_r := \{x \in X : f(x) \leq_H rb\}$ is closed for each $r \in \mathbb{R}$.

(ii) ([36]) f is said to be (b,H)-lower semicontinuous from above ((b,H)-lsca) at $x \in X$ if, for each $r \in \mathbb{R}$ and $x_m \to x$, from $f(x_1) + rb \leq_H 0$ and $f(x_{m+1}) + t_m b \leq_H f(x_m)$, $t_m \geq 0$, for all $m \in \mathbb{N}$, it follows that $f(x) + rb \leq_H 0$.

(iii) In [17,25,28,36,45], f is termed H-sequentially lower monotone (H-slm) at $x \in X$ if from $x_m \to x$ such that $f(x_{m+1}) \leq_H f(x_m)$ for all m, one has $f(x) \leq_H f(x_m)$ for all m.

Remark 7 (b,H)-lower semicontinuity can be restated as follows: f is called (b,H)-lsc at $x \in X$ if for all $r \in \mathbb{R}$ and $x_m \to x$ such that $f(x_m) + rb \leq_H 0$, one has $f(x) + rb \leq_H 0$.

Proposition 1 *Let X, H, b, and f be as in Definition 3.*

(i) If f is H-lsc, then f is both H-slm and H-qlsca.

(ii) *If f is H-slm, then f is also (b,H)-lsca for every $b \in H \setminus \{0\}$. When $n = 1$, the last two notions are equivalent.*
(iii) *For $n = 1$, f is $\mathbb{R}_+$-qlsca if and only if f is strictly-decreasingly lower semicontinuous.*
(iv) *For $n = 1$ and $b > 0$, $(b,\mathbb{R}_+)$-lower semicontinuity from above is properly stronger than $(b,\mathbb{R}_+)$-quasi lower semicontinuity from above.*

Proof (i) Let f be H-lsc, i.e., M_r is closed for all $r \in \mathbb{R}$, and let $b \in \mathbb{R}^n \setminus \{0\}$ and $x \in X$. Assume that $x_m \to x$ and $f(x_{m+1}) \leq_H f(x_m)$ for all m. Fixing m and choosing $rb = -f(x_m)$, one has $f(x_{m+k}) + rb \leq_H f(x_{m+1}) + rb \leq_H 0$ for all $k \geq 1$. Hence, $x_{m+k} \in M_r$ and tends to x as $k \to \infty$. Consequently, $x \in M_r$, i.e., $f(x) \leq_H -rb = f(x_m)$. As m is arbitrary, f is H$-slm$ at x. We prove that f is H-qlsca at x. Assume that $x_m \to x$ and $f(x_m) + rb \leq_H 0$, $f(x_m) \nleq_H f(x_{m+1})$ for all m. Then, by definition, from only the first two just assumed conditions and the closedness of M_r, one has $f(x) + rb \leq 0$, i.e., f is H-qlsca.

(ii) Let $x \in X$ and suppose that f is H-slm at x. Consider $b \in H \setminus \{0\}$, $r \in \mathbb{R}$ and $x_m \to x$ such that $f(x_1) + rb \leq_H 0$ and $f(x_{m+1}) + t_m b \leq_H f(x_m)$, $t_m \geq 0$, for all $m \in \mathbb{N}$. Then,

$$f(x_{m+1}) - f(x_m) \in -t_m b - H \subset -H - H \subset -H,$$

i.e., $f(x_{m+1}) \leq_H f(x_m)$ for all m. As f is H-slm at x, one has $f(x) \leq_H f(x_m)$ for all m. Hence, $f(x) + rb \leq_H f(x_m) + rb \leq_H f(x_1) + rb \leq_H 0$, and so f is (b,H)-lsca at x. The assertion for the case $n = 1$ is easily checked directly.

(iii) Let $x \in X$. To see the “only if” part, let $x_m \to x$ and $f(x_{m+1}) < f(x_m)$ for all m. Then, fixing $b \neq 0$ and $m \in \mathbb{N}$, and taking $r = -\dfrac{f(x_m)}{b}$, one has $f(x_{m+k}) + rb = f(x_{m+k}) - f(x_m) < 0$. As $x_{m+k} \to x$ and f is $(b,\mathbb{R}_+)$-qlsca at x, one has $f(x) + rb = f(x) - f(x_m) \leq 0$ for all m. Hence, f is strictly-decreasingly lower semicontinuous at x. Now we verify the “if” part. Let $b \neq 0$, $r \in \mathbb{R}$ be arbitrary and $x_m \to x$ with $f(x_m) + rb \leq 0$ and $f(x_{m+1}) < f(x_m)$ for all m. As f is strictly-decreasingly lower semicontinuous at x, $f(x) \leq f(x_m)$ for all m, so for any $m \in \mathbb{N}$ we have $f(x) + rb \leq f(x_m) + rb \leq 0$, and the proof of this part is finished.

(iv) Let $b > 0$, $x \in X$ and suppose that f is (b,H)-lsca at x. To derive $(b,\mathbb{R}_+)$-quasi lower semicontinuity from above at x, let $x_m \to x$ such that $f(x_m) + rb \leq 0$ and $f(x_{m+1}) < f(x_m)$ for all m. Then, since f is (b,H)-lsca we have $f(x) + rb \leq 0$, so f is (b,H)-qlsca. For the properness of this implication, see [1, Example 7] to see a case that f is $(b,\mathbb{R}_+)$-qlsca but not $(b,\mathbb{R}_+)$-lsca.

For $n > 1$, in the next example we prove that the (b,H)-qlsca notion is not stronger than the (b,H)-lsca concept.

Example 4 Let $H = \mathbb{R}^2_+$ and $f : \mathbb{R} \to \mathbb{R}^2$ be defined by

$$f(x) = \begin{cases} (x,0) & \text{if } x \geq 0, \\ (-1,-1) & \text{if } x < 0. \end{cases}$$

Then, for any $b \in \mathbb{R}^2 \setminus \{(0,0)\}$, f is $(b,\mathbb{R}^2_+)$-qlsca at 0. Indeed, assume that $r \in \mathbb{R}$ and $x_m \to 0$ such that $f(x_m) + rb \leq_{\mathbb{R}^2_+} 0$ and $f(x_m) \not\leq_{\mathbb{R}^2_+} f(x_{m+1})$ for all m. There are the following two cases.

- Case 1: there exists $m_0 \in \mathbb{N}$ such that $x_{m_0} < 0$. Then, if $x_{m_0+1} \geq 0$ then $f(x_{m_0}) - f(x_{m_0+1}) = (-1,-1) - (x_{m_0+1},0) = (-1-x_{m_0+1},-1) \in \mathbb{R}^2_-$, which is impossible. If $x_{m_0+1} < 0$ then $f(x_{m_0}) - f(x_{m_0+1}) = (-1,-1) - (-1,-1) = (0,0)$, which is impossible as well.
- Case 2: $x_m \geq 0$ for all m. Then, $(x_m,0) - (x_{m+1},0) \notin -\mathbb{R}^2_+$ means that $x_{m+1} < x_m$. Hence, $0 < x_{m+1} < x_m$ for all m (because $x_{m_0} = 0$ for some m_0 implies $x_{m+1} = x_m = 0$). So, $f(x_m) - f(0) = (x_m,0) \in \mathbb{R}^2_+ \setminus \{(0,0)\}$ for all m. Therefore, $f(0) + rb \leq_{\mathbb{R}^2_+} f(x_m) + rb \leq_{\mathbb{R}^2_+} 0$, i.e., f is $(b,\mathbb{R}^2_+)$-qlsca at 0.

However, f is not $(b,\mathbb{R}^2_+)$-lsca at 0. Indeed, taking $x_m = -1/m$, one finds $r \in \mathbb{R}$ such that

$$\begin{cases} f(x_1) + rb = (-1,-1) + rb \leq_{\mathbb{R}^2_+} 0, \\ f(0) + rb = rb \in \mathbb{R}^2_+ \setminus \{(0,0)\}. \end{cases}$$

Moreover, choosing $t_m = 0$, one has $f(x_{m+1}) + t_m b = (-1,-1) \leq_{\mathbb{R}^2_+} f(x_m)$. Observe that if f is $(b,\mathbb{R}^2_+)$-lsca at 0, then by definition, $f(0) + rb = rb \leq_{\mathbb{R}^2_+} 0$, which is a contradiction.

From now on, we apply Definition 3 only for special cases of $b \in H \setminus \{0\}$, not generally $b \in \mathbb{R}^n \setminus \{0\}$. In the following theorem, we provide a vector variant of the EVP for Henig (C,ε)-proper efficient solutions of $(\mathscr{VEP})$ through scalarization, by using the functional φ_{D_ρ,y_0}, with $\rho > 0$, $y_0 \in D\setminus\{0\}$, and $C \in \mathscr{A}$, and by assuming that $F(x_0,\cdot)$ is (y_0,D_ρ)-qlsca for all $\rho > 0$.

Theorem 3 *Let $C \in \mathscr{A}$, $\varepsilon > 0$, $x_0 \in X$, and $y_0 \in D\setminus\{0\}$. Suppose that F satisfies the diagonal null property, and $F(x_0,\cdot)$ is (y_0,D_ρ)-qlsca for all $\rho > 0$. If $x_0 \in \mathrm{He}(F,X,C,\varepsilon)$, then there exists $\bar{\rho} > 0$ such that for each $\lambda > 0$ there exists $x_\lambda \in X$ satisfying*

(i) $d(x_\lambda,x_0) \leq \lambda$;

(ii) $\displaystyle\max_{i\in\{1,2,\ldots,p\}} \left\{ \frac{\alpha_i \cdot F(x_0,x_\lambda)}{\alpha_i \cdot y_0} \right\} \leq -k_\lambda d(x_\lambda,x_0)$;

(iii) $\displaystyle\max_{i\in\{1,2,\ldots,p\}} \left\{ \frac{\alpha_i \cdot F(x_0,x_\lambda)}{\alpha_i \cdot y_0} \right\} < \max_{i\in\{1,2,\ldots,p\}} \left\{ \frac{\alpha_i \cdot F(x_0,x)}{\alpha_i \cdot y_0} \right\} + k_\lambda d(x,x_\lambda), \forall x \neq x_\lambda.$

If additionally the hypothesis $(\mathscr{A}_{D_{\bar{\rho}}})$ holds, then

(iv) $\displaystyle 0 \leq \max_{i\in\{1,2,\ldots,p\}} \left\{ \frac{\alpha_i \cdot F(x_\lambda,x)}{\alpha_i \cdot y_0} \right\} + k_\lambda d(x_\lambda,x)$ *for all* $x \in X$,

$\displaystyle 0 < \max_{i\in\{1,2,\ldots,p\}} \left\{ \frac{\alpha_i \cdot F(x_\lambda,x)}{\alpha_i \cdot y_0} \right\} + k_\lambda d(x_\lambda,x)$ *for all* $x \in X$ *such that* $F(x_\lambda,x) + k_\lambda d(x_\lambda,x) y_0 \neq 0$,

where $k_\lambda := \dfrac{\varepsilon}{\lambda} \inf_{c\in C} \varphi_{D_{\bar{\rho}},y_0}(c)$ *and* $\alpha_i := (\bar{\rho} u + e_i)A$ *for all* $i \in \{1,2,\ldots,p\}$.

Proof Since $C \in \mathscr{A}$, there exists $\hat{\rho}, r > 0$ such that $C \in \mathscr{A}_{\hat{\rho},r}$. Moreover, as $x_0 \in \mathrm{He}(F,X,C,\varepsilon)$, by Remark 2 we know that there exists $\bar{\rho} > 0$ (that we can assume to be lower than or equal to $\hat{\rho}$) such that

$$(F(x_0,X) + C(\varepsilon)) \cap \left(-\operatorname{int} D_{\bar{\rho}}\right) = \emptyset. \tag{10}$$

By statements (7) and (10) we deduce that $\varphi_{D_{\bar{\rho}},y_0}(F(x_0,x)+c) \geq 0$, for all $x \in X$, $c \in C(\varepsilon)$. Since $\varphi_{D_{\bar{\rho}},y_0}$ is subadditive, it follows that

$$0 \leq \varphi_{D_{\bar{\rho}},y_0}(F(x_0,x)+c) \leq \varphi_{D_{\bar{\rho}},y_0}(F(x_0,x)) + \varphi_{D_{\bar{\rho}},y_0}(c),\ \forall x \in X,\ \forall c \in C(\varepsilon). \tag{11}$$

By Remark 6(a), we know that $\inf\limits_{c \in C} \varphi_{D_{\bar{\rho}},y_0}(c) \geq r > 0$ for all $c \in C$, and since $\varphi_{D_{\bar{\rho}},y_0}$ is positively homogeneous, it follows that

$$\inf_{c \in C(\varepsilon)} \varphi_{D_{\bar{\rho}},y_0}(c) = \varepsilon \inf_{c \in C} \varphi_{D_{\bar{\rho}},y_0}(c) > 0.$$

Thus, from (11) we have that

$$0 \leq \varphi_{D_{\bar{\rho}},y_0}(F(x_0,x)) + \varepsilon \inf_{c \in C} \varphi_{D_{\bar{\rho}},y_0}(c),\ \forall x \in X. \tag{12}$$

We define $h(x) := \varphi_{D_{\bar{\rho}},y_0}(F(x_0,x))$ and claim that h is strictly-decreasingly lower semicontinuous. Let $x_m \to \hat{x}$ be such that $h(x_{m+1}) < h(x_m)$ for all $m \in \mathbb{N}$. Then, $F(x_0,x_m) \not\leq_{D_{\bar{\rho}}} F(x_0,x_{m+1})$. For fixed $m \geq 1$, the sequences $(z_k) := (x_{m+k})_k$ satisfies the conditions

$$\begin{cases} F(x_0,z_k) \not\leq_{D_{\bar{\rho}}} F(x_0,z_{k+1}), \\ \lim\limits_{k\to\infty} z_k = \lim\limits_{m\to\infty} x_m = \hat{x}. \end{cases}$$

For $r = -h(x_m)$, $F(x_0,z_k) + ry_0 \leq_{D_{\bar{\rho}}} 0$ for all k. Indeed, supposing the contrary and using (9), we have $h(x_{m+k})$-$h(x_m) = \varphi_{D_{\bar{\rho}},y_0}(F(x_0,z_k) - h(x_m)y_0) > 0$, and so $h(x_{m+k}) > h(x_m)$, which is impossible.

As $F(x_0,\cdot)$ is $(y_0,D_{\bar{\rho}})$-qlsca, $F(x_0,\hat{x}) - h(x_m)y_0 \leq_{D_{\bar{\rho}}} 0$ and hence $h(\hat{x}) \leq h(x_m)$ for $m \in \mathbb{N}$, i.e., h is strictly-decreasingly lower semicontinuous. Furthermore, from (12) it follows that $-\varepsilon \inf\limits_{c \in C} \varphi_{D_{\bar{\rho}},y_0}(c) \leq h(x)$ for all $x \in X$, and so h is bounded from below. Actually, since F verifies the diagonal null property, we have that $h(x_0) = 0$ and then

$$h(x_0) \leq h(x) + \varepsilon \inf_{c \in C} \varphi_{D_{\bar{\rho}},y_0}(c),\ \forall x \in X.$$

Thus, function h satisfies the hypotheses of Lemma 3, from which we know that for each $\lambda \in \mathbb{R}$ there exists $x_\lambda \in X$ verifying

(a) $d(x_\lambda,x_0) \leq \lambda$;
(b) $h(x_\lambda) \leq h(x_0) - k_\lambda d(x_\lambda,x_0)$;
(c) $h(x) + k_\lambda d(x,x_\lambda) > h(x_\lambda),\ \forall x \neq x_\lambda$.

So (i) is satisfied. By Lemma 4, statement (b) is equivalent to

$$\max_{i\in\{1,2,\ldots,p\}}\left\{\frac{\alpha_i\cdot F(x_0,x_\lambda)}{\alpha_i\cdot y_0}\right\}\leq 0-k_\lambda d(x_\lambda,x_0),$$

and hence (ii) also holds.

Property (iii) is proved directly by applying Lemma 4 to statement (c).

Finally, assume that F verifies the hypothesis $(\mathscr{A}_{D_{\bar\rho}})$ and suppose that for some $\lambda>0$ there exists $\bar x\in X$ such that

$$F(x_\lambda,\bar x)+k_\lambda d(\bar x,x_\lambda)y_0\in -D_{\bar\rho}\backslash\{0\}.$$

Clearly, $\bar x\neq x_\lambda$. By statement (b), we deduce that $F(x_0,x_\lambda)\in -D_{\bar\rho}$ and then, by virtue of the hypothesis $(\mathscr{A}_{D_{\bar\rho}})$,

$$F(x_0,\bar x)\in F(x_0,x_\lambda)+F(x_\lambda,\bar x)-D_{\bar\rho}.$$

Thus,

$$F(x_0,\bar x)+k_\lambda d(\bar x,x_\lambda)y_0\in F(x_0,x_\lambda)-D_{\bar\rho}\backslash\{0\}.$$

By the $D_{\bar\rho}$-monotonicity of $\varphi_{D_{\bar\rho},y_0}$ and statement (9), it follows that

$$h(\bar x)+k_\lambda d(\bar x,x_\lambda)=\varphi_{D_{\bar\rho},y_0}(F(x_0,\bar x)+k_\lambda d(\bar x,x_\lambda)y_0)\leq\varphi_{D_{\bar\rho},y_0}(F(x_0,x_\lambda))=h(x_\lambda),$$

which contradicts (c). Therefore,

$$F(x_\lambda,x)+k_\lambda d(x,x_\lambda)y_0\notin -D_{\bar\rho}\backslash\{0\},\ \forall x\in X.$$

Applying Lemma 4 and properties (7)-(9) to the statement above, we conclude that (iv) is also verified. The proof is complete.

Remark 8 If we define $\overline{F}_\lambda(x,y):=F(x,y)+k_\lambda d(x,y)y_0$, then statement (iv) of Theorem 3 means that $\overline{F}_\lambda(x_\lambda,x)\cap(-\operatorname{int}D_{\bar\rho})=\emptyset$, so $x_\lambda\in\operatorname{He}(\overline{F}_\lambda,X,D)$.

5 An EVP for cone-constrained vector equilibrium problems

In this section, we pay our attention to the following constrained VEP

$$\text{Find } x_0\in S \text{ such that } F(x_0,S)\cap(-D\backslash\{0\})=\emptyset, \qquad (\mathscr{CVEP})$$

where $S:=\{x\in X: g(x)\in -K\}$, $g:X\to\mathbb{R}^l$ and K is the polyhedral cone $\{z\in\mathbb{R}^l: Bz^t\in\mathbb{R}^q_+\}$ with $B\in\mathscr{M}_{q\times l}$. We suppose that K is solid.

We aim to provide a version of the EVP for Henig (C,ε)-proper solutions of $(\mathscr{CVEP})$. Let $G:X\times X\to\mathbb{R}^l$ be defined as $G(x,y)=g(y)-g(x)$.

Lemma 5 *Let $\varepsilon\geq 0$, $C\in\overline{\mathscr{H}}$, $x_0\in S$, and $\rho>0$. If $F(x_0,S)\cap(-C(\varepsilon)-\operatorname{int}D_\rho)=\emptyset$, then*

$$[(F,G)(x_0,X)+(C(\varepsilon),g(x_0))]\cap[-\operatorname{int}(D_\rho\times K)]=\emptyset. \qquad (13)$$

Proof Suppose by reasoning to the contrary that (13) does not hold. Then, there exist $\bar{x} \in X$ and $c \in C(\varepsilon)$ such that

$$(F,G)(x_0,\bar{x}) + (c, g(x_0)) \in (-\operatorname{int} D_\rho) \times (-\operatorname{int} K).$$

Thus,

$$F(x_0,\bar{x}) \in -c - \operatorname{int} D_\rho \subset -C(\varepsilon) - \operatorname{int} D_\rho,$$
$$G(x_0,\bar{x}) + g(x_0) \in -\operatorname{int} K.$$

The last inclusion means that $g(\bar{x}) \in -\operatorname{int} K$. So, $\bar{x} \in S$ and $F(x_0,\bar{x}) \in -C(\varepsilon) - \operatorname{int} D_\rho$, which is a contradiction.

For $\rho > 0$, let us denote $H_\rho := D_\rho \times K \subset \mathbb{R}^n \times \mathbb{R}^l$. The next lemma was proved in [22, Lemma 3.9].

Lemma 6 *Let $\rho > 0$ and $(y_0, z_0) \in D \setminus \{0\} \times \operatorname{int} K$. Then,*

$$\varphi_{H_\rho,(y_0,z_0)}(y,z) = \max\left\{ \max_{i\in\{1,2,\ldots,p\}} \left\{ \frac{\alpha_i \cdot y}{\alpha_i \cdot y_0} \right\}, \max_{j\in\{1,2,\ldots,q\}} \left\{ \frac{\beta_j \cdot z}{\beta_j \cdot z_0} \right\} \right\},$$

for all $(y,z) \in \mathbb{R}^n \times \mathbb{R}^l$, where $\alpha_i := (\rho u + e_i)A$, $i \in \{1,2,\ldots,p\}$ and β_j is the jth row of B for $j \in \{1,2,\ldots,q\}$.

Remark 9 H_ρ is a closed convex pointed cone in $\mathbb{R}^n \times \mathbb{R}^l$. So, for $(y_0,z_0) \in D\setminus\{0\} \times \operatorname{int} K \subset \operatorname{int} H_\rho$, it is clear that $\varphi_{H_\rho,(y_0,z_0)}$ satisfies the same properties as φ_{D_ρ,y_0} in its domain of definition (see Remark 5). For the convenience of the reader, we remind the following formulae for $\varphi_{H_\rho,(y_0,z_0)}$, which can be directly verified,

$$\{(y,z) \in \mathbb{R}^n \times \mathbb{R}^l : \varphi_{H_\rho,(y_0,z_0)}(y,z) < r\} = r(y_0,z_0) - \operatorname{int} H_\rho, \tag{14}$$
$$\{(y,z) \in \mathbb{R}^n \times \mathbb{R}^l : \varphi_{H_\rho,(y_0,z_0)}(y,z) = r\} = r(y_0,z_0) - \operatorname{bd} H_\rho,$$
$$\varphi_{H_\rho,(y_0,z_0)}(y + ry_0, z + rz_0) = \varphi_{H_\rho,(y_0,z_0)}(y,z) + r,\ \forall (y,r) \in \mathbb{R}^n \times \mathbb{R}^l.$$

Theorem 4 *Let $C \in \mathscr{A}$, $\varepsilon > 0$, $x_0 \in S$, and $(y_0,z_0) \in D\setminus\{0\} \times \operatorname{int} K$. Suppose that F satisfies the diagonal null property on S, $F(x_0,\cdot)$ is (y_0, D_ρ)-qlsca for all $\rho > 0$, and g is (z_0,K)-qlsca. If $x_0 \in \operatorname{He}(F,S,C,\varepsilon)$, then there exists $\bar{\rho} > 0$ such that for each $\lambda > 0$, there exists $x_\lambda \in S$ verifying*

(i) $d(x_\lambda, x_0) \leq \lambda$;

(ii) $\max\left\{ \max_{i\in\{1,2,\ldots,p\}} \left\{ \frac{\alpha_i \cdot F(x_0,x_\lambda)}{\alpha_i \cdot y_0} \right\}, \max_{j\in\{1,2,\ldots,q\}} \left\{ \frac{\beta_j \cdot g(x_\lambda)}{\beta_j \cdot z_0} \right\} \right\} \leq -k_\lambda d(x_\lambda, x_0)$;

(iii) $\max\left\{ \max_{i\in\{1,2,\ldots,p\}} \left\{ \frac{\alpha_i \cdot F(x_0,x_\lambda)}{\alpha_i \cdot y_0} \right\}, \max_{j\in\{1,2,\ldots,q\}} \left\{ \frac{\beta_j \cdot g(x_\lambda)}{\beta_j \cdot z_0} \right\} \right\}$
$< \max\left\{ \max_{i\in\{1,2,\ldots,p\}} \left\{ \frac{\alpha_i \cdot F(x_0,x)}{\alpha_i \cdot y_0} \right\}, \max_{j\in\{1,2,\ldots,q\}} \left\{ \frac{\beta_j \cdot g(x)}{\beta_j \cdot z_0} \right\} \right\} + k_\lambda d(x, x_\lambda)$,
$\forall x \neq x_\lambda$.

If additionally the hypothesis ($\mathscr{A}_{D_{\bar{\rho}}}$) holds, then for all $x \in X$

(iv) $0 \leq \max\left\{\max_{i\in\{1,2,\ldots,p\}}\left\{\frac{\alpha_i \cdot F(x_\lambda,x)}{\alpha_i \cdot y_0}\right\}, \max_{j\in\{1,2,\ldots,q\}}\left\{\frac{\beta_j \cdot (g(x)-g(x_\lambda))}{\beta_j \cdot z_0}\right\}\right\} + k_\lambda d(x_\lambda,x)$, *being the inequality strict whenever* $(F,G)(x_\lambda,x)+k_\lambda d(x_\lambda,x)(y_0,z_0) \neq (0,0)$,

where $k_\lambda := \dfrac{\varepsilon}{\lambda}\inf_{c\in C}\varphi_{D_{\bar\rho},y_0}(c)$, $\alpha_i := (\bar\rho u + e_i)A$, *for all* $i \in \{1,2,\ldots,p\}$ *and* β_j *is the jth row of* B *for* $j \in \{1,2,\ldots,q\}$.

Proof Since $C \in \mathscr{A}$, there exist $\hat\rho, r > 0$ such that $C \in \mathscr{A}_{\hat\rho,r}$. Moreover, as $x_0 \in \mathrm{He}(F,S,C,\varepsilon)$, by the definition and Lemma 5 there exists $\bar\rho > 0$ (that can be considered less than or equal to $\hat\rho$) such that

$$[(F,G)(x_0,X)+(C(\varepsilon),g(x_0))] \cap (-\operatorname{int} H_{\bar\rho}) = \emptyset.$$

Thus, by statement (14) we deduce that

$$\varphi_{H_{\bar\rho},(y_0,z_0)}((F,G)(x_0,x)+(c,g(x_0))) \geq 0,\ \forall x \in X,\ \forall c \in C(\varepsilon).$$

By the subadditivity of $\varphi_{H_{\bar\rho},(y_0,z_0)}$ we have, for all $x \in X$ and $c \in C(\varepsilon)$,

$$\begin{aligned} 0 &\leq \varphi_{H_{\bar\rho},(y_0,z_0)}((F,G)(x_0,x)+(0,g(x_0))+(c,0)) \\ &\leq \varphi_{H_{\bar\rho},(y_0,z_0)}((F,G)(x_0,x)+(0,g(x_0))) + \varphi_{H_{\bar\rho},(y_0,z_0)}(c,0). \end{aligned}$$

Since $C \in \mathscr{A}$, by Lemma 6 it is clear that $\varphi_{H_{\bar\rho},(y_0,z_0)}(c,0) = \varphi_{D_{\bar\rho},y_0}(c)$ for all $c \in C(\varepsilon)$. and then

$$0 \leq \varphi_{H_{\bar\rho},(y_0,z_0)}((F,G)(x_0,x)+(0,g(x_0))) + \varepsilon \inf_{c\in C}\varphi_{D_{\bar\rho},y_0}(c),\ \forall x \in X.$$

Define

$$h(x) := \varphi_{H_{\bar\rho},(y_0,z_0)}((F,G)(x_0,x)+(0,g(x_0))) = \varphi_{H_{\bar\rho},(y_0,z_0)}(F(x_0,x),g(x)).$$

Then, following a reasoning similar to the proof of Theorem 3 but for this new function h depending on the couple (F,G), we conclude that h is strictly-decreasingly lower semicontinuous. Also, since F satisfies the diagonal null property on S and $g(x_0) \in -K$, by Lemma 6 we have

$$\begin{aligned} h(x_0) &= \varphi_{H_{\bar\rho},(y_0,z_0)}(F(x_0,x_0),g(x_0)) = \varphi_{H_{\bar\rho},(y_0,z_0)}(0,g(x_0)) \\ &= \max\{0, \varphi_{K,z_0}(g(x_0))\} = 0. \end{aligned}$$

Thus, $h(x_0) \leq h(x) + \varepsilon \inf_{c\in C}\varphi_{D_{\bar\rho},y_0}(c)$, $\forall x \in X$, and by Lemma 3 we know that for each $\lambda > 0$, there exists $x_\lambda \in X$ such that conditions (a)-(c) in the proof of Theorem 3 are satisfied for this new function h. Then, (i) holds. Also by Lemma 6, statement (b) is equivalent to

$$\begin{aligned} \varphi_{H_{\bar\rho},(y_0,z_0)}((F,G)(x_0,x_\lambda)+(0,g(x_0))) &= \varphi_{H_{\bar\rho},(y_0,z_0)}(F(x_0,x_\lambda),g(x_\lambda)) \\ &= \max\left\{\varphi_{D_{\bar\rho},y_0}(F(x_0,x_\lambda)), \varphi_{K,z_0}(g(x_\lambda))\right\} \\ &\leq -k_\lambda d(x_\lambda,x_0). \end{aligned}$$

Hence, (ii) holds. Moreover, from the last inequality above we deduce in particular that $\varphi_{K,z_0}(g(x_\lambda)) \leq 0$, from which we have $g(x_\lambda) \in -K$, which means that $x_\lambda \in S$. Statement (iii) follows directly by applying Lemma 6 to statement (c).

Finally, assume that F verifies the hypothesis $(\mathscr{A}_{D_{\bar{\rho}}})$ and suppose that for $\lambda > 0$ there exists $\bar{x} \in X$ such that

$$(F,G)(x_\lambda,\bar{x}) + k_\lambda d(\bar{x},x_\lambda)(y_0,z_0) \in -H_{\bar{\rho}} \setminus \{(0,0)\}.$$

This means that

$$\begin{aligned} F(x_\lambda,\bar{x}) + k_\lambda d(\bar{x},x_\lambda) y_0 &\in -D_{\bar{\rho}}, \\ G(x_\lambda,\bar{x}) + k_\lambda d(\bar{x},x_\lambda) z_0 &\in -K, \end{aligned}$$

and either $F(x_\lambda,\bar{x}) + k_\lambda d(\bar{x},x_\lambda) y_0 \neq 0$ or $G(x_\lambda,\bar{x}) + k_\lambda d(\bar{x},x_\lambda) z_0 \neq 0$. Then, with the same reasoning as in the proof of Theorem 3, the above implies that $\bar{x} \neq x_\lambda$ and

$$F(x_0,\bar{x}) + k_\lambda d(\bar{x},x_\lambda) y_0 \in F(x_0,x_\lambda) - D_{\bar{\rho}}. \tag{15}$$

Also,

$$\begin{aligned} G(x_0,\bar{x}) + k_\lambda d(\bar{x},x_\lambda) z_0 &= g(\bar{x}) - g(x_\lambda) + g(x_\lambda) - g(x_0) + k_\lambda d(\bar{x},x_\lambda) z_0 \\ &= G(x_\lambda,\bar{x}) + G(x_0,x_\lambda) + k_\lambda d(\bar{x},x_\lambda) z_0 \\ &\in G(x_0,x_\lambda) - K. \end{aligned} \tag{16}$$

Hence, by (15) and (16), we derive that

$$\begin{aligned} &(F,G)(x_0,\bar{x}) + (0,g(x_0)) + k_\lambda d(\bar{x},x_\lambda)(y_0,z_0) \in \\ &(F,G)(x_0,x_\lambda) + (0,g(x_0)) - H_{\bar{\rho}}. \end{aligned}$$

Consequently, by the properties of functional $\varphi_{H_{\bar{\rho}},(y_0,z_0)}$,

$$h(\bar{x}) + k_\lambda d(\bar{x},x_\lambda) \leq h(x_\lambda),$$

which contradicts property (c) of h. Hence,

$$(F,G)(x_\lambda,x) + k_\lambda d(x,x_\lambda)(y_0,z_0) \notin -H_{\bar{\rho}} \setminus \{(0,0)\}, \; \forall x \in X,$$

and so $\varphi_{H_{\bar{\rho}},(y_0,z_0)}(F(x_\lambda,x), G(x_\lambda,x)) + k_\lambda d(x,x_\lambda) \geq 0$ for all $x \in X$, where the inequality is strict whenever $(F,G)(x_\lambda,x) + k_\lambda d(x,x_\lambda)(y_0,z_0) \neq (0,0)$. Thus, part (iv) follows directly by Lemma 6, and the proof is complete.

6 Applications to multiobjective optimization and variational inequalities

In this section, we are going to apply the results stated in Sections 4 and 5, through the solutions of a related vector variational inequality problem, to the special case of a general multiobjective optimization problem.

Consider a function $f : X \to \mathbb{R}^n$ and the following multiobjective optimization problem

$$\text{Minimize } f(x) \text{ subject to } x \in M, \qquad (\mathscr{MOP})$$

where $\emptyset \neq M \subset X$ is the feasible set.

We focus on the study of (C,ε)-proper efficient solutions of $(\mathscr{MOP})$, which are defined in the following way (see [21]).

Definition 5 Let $\varepsilon \geq 0$ and $C \in \overline{\mathscr{H}}$. A point $x_0 \in M$ is a Henig (C,ε)-proper efficient solution of $(\mathscr{MOP})$, and we denote it by $x_0 \in \mathrm{He}_{\mathscr{MOP}}(f,M,C,\varepsilon)$, if there exists $D' \in \mathscr{G}(C)$ such that

$$(f(M) - f(x_0)) \cap (-C(\varepsilon) - \operatorname{int} D') = \emptyset.$$

If $C = D \backslash \{0\}$, then we obtain the notion of an exact Henig proper efficient solution.

Remark 10 Let $F(x,y) := f(y) - f(x)$. It is clear that $x_0 \in \mathrm{He}_{\mathscr{MOP}}(f,X,C,\varepsilon)$ if and only if x_0 is a (C,ε)-proper efficient solution of $(\mathscr{VEP})$ and $x_0 \in \mathrm{He}_{\mathscr{MOP}}(f,S,C,\varepsilon)$ if and only if x_0 is a (C,ε)-proper efficient solution of $(\mathscr{CVEP})$.

The next two results provide versions of the EVP for Henig (C,ε)-proper efficient solutions of $(\mathscr{MOP})$ for the unconstrained case (i.e., when $M = X$) and also for $M = S$. They are direct consequences of Remark 10 and Theorems 3 and 4, respectively.

Corollary 1 *Let $C \in \mathscr{A}$, $\varepsilon > 0$, $x_0 \in X$, and $y_0 \in D\backslash\{0\}$. Suppose that $f - f(x_0)$ is (y_0, D_ρ)-qlsca for all $\rho > 0$. If $x_0 \in \mathrm{He}_{\mathscr{MOP}}(f,X,C,\varepsilon)$, then there exists $\bar{\rho} > 0$ such that for each $\lambda > 0$, there exists $x_\lambda \in X$ satisfying*

(i) $d(x_\lambda, x_0) \leq \lambda$;

(ii) $\displaystyle\max_{i\in\{1,2,\ldots,p\}} \left\{ \frac{\alpha_i \cdot (f(x_\lambda) - f(x_0))}{\alpha_i \cdot y_0} \right\} \leq -k_\lambda d(x_\lambda, x_0)$;

(iii) $\displaystyle\max_{i\in\{1,2,\ldots,p\}} \left\{ \frac{\alpha_i \cdot (f(x_\lambda) - f(x_0))}{\alpha_i \cdot y_0} \right\} < \max_{i\in\{1,2,\ldots,p\}} \left\{ \frac{\alpha_i \cdot (f(x) - f(x_0))}{\alpha_i \cdot y_0} \right\} + k_\lambda d(x, x_\lambda)$, $\forall x \neq x_\lambda$.

(iv) $\displaystyle 0 \leq \max_{i\in\{1,2,\ldots,p\}} \left\{ \frac{\alpha_i \cdot (f(x) - f(x_\lambda))}{\alpha_i \cdot y_0} \right\} + k_\lambda d(x_\lambda, x)$ *for all* $x \in X$,
$\displaystyle 0 < \max_{i\in\{1,2,\ldots,p\}} \left\{ \frac{\alpha_i \cdot (f(x) - f(x_\lambda))}{\alpha_i \cdot y_0} \right\} + k_\lambda d(x_\lambda, x)$ *for all* $x \in X$ *such that* $f(x) - f(x_\lambda) + k_\lambda d(x_\lambda, x) y_0 \neq 0$,

where $k_\lambda := \dfrac{\varepsilon}{\lambda} \inf_{c\in C} \varphi_{D_{\bar{\rho}}, y_0}(c)$ *and* $\alpha_i := (\bar{\rho} u + e_i) A$ *for all* $i \in \{1,2,\ldots,p\}$.

Remark 11 (a) Let us note that if in Theorem 1 we consider $C = y_0 + D$, then we obtain necessary conditions for approximate proper efficient solutions in the sense of El Maghri (see [10]), where in this case $k_\lambda = \varepsilon/\lambda$ for each $\lambda \in \mathbb{R}$.

(b) Statement (iv) of Theorem 1 implies that $x_\lambda \in \mathrm{He}_{\mathscr{MOP}}(f_\lambda, X, D)$, where $f_\lambda(\cdot) := f(\cdot) + k_\lambda d(\cdot, x_\lambda) y_0$.

(c) In [25, Section 5] the authors provided necessary conditions for several types of approximate solutions through a version of the EVP, with a motivation similar to that of the results presented in this section. Notice that in Theorem 1, we provide an alternative improved version of the EVP for approximate proper solutions of $(\mathscr{MOP})$ and for more general sets C than in [25, Section 5].

Moreover, in the next result we also study the case in which the feasible set is given by a cone constraint.

Corollary 2 *Let $C \in \mathscr{A}$, $\varepsilon > 0$, $x_0 \in S$, $(y_0, z_0) \in D\backslash\{0\} \times \operatorname{int} K$. Suppose that $f - f(x_0)$ is (y_0, D_ρ)-qlsca for all $\rho > 0$, and g is (z_0, K)-qlsca. If $x_0 \in \mathrm{He}_{\mathscr{MOP}}(f, S, C, \varepsilon)$, then there exists $\bar{\rho} > 0$ such that for each $\lambda > 0$ there exists $x_\lambda \in S$ verifying*

(i) $d(x_\lambda, x_0) \le \lambda$;

(ii) $\max\left\{\max\limits_{i\in\{1,2,\ldots,p\}}\left\{\dfrac{\alpha_i\cdot(f(x_\lambda)-f(x_0)}{\alpha_i\cdot y_0}\right\}, \max\limits_{j\in\{1,2,\ldots,q\}}\left\{\dfrac{\beta_j\cdot g(x_\lambda)}{\beta_j\cdot z_0}\right\}\right\} \le -k_\lambda d(x_\lambda, x_0)$;

(iii) $\max\left\{\max\limits_{i\in\{1,2,\ldots,p\}}\left\{\dfrac{\alpha_i\cdot(f(x_\lambda)-f(x_0))}{\alpha_i\cdot y_0}\right\}, \max\limits_{j\in\{1,2,\ldots,q\}}\left\{\dfrac{\beta_j\cdot g(x_\lambda)}{\beta_j\cdot z_0}\right\}\right\}$
$< \max\left\{\max\limits_{i\in\{1,2,\ldots,p\}}\left\{\dfrac{\alpha_i\cdot(f(x)-f(x_0))}{\alpha_i\cdot y_0}\right\}, \max\limits_{j\in\{1,2,\ldots,q\}}\left\{\dfrac{\beta_j\cdot g(x)}{\beta_j\cdot z_0}\right\}\right\} + k_\lambda d(x, x_\lambda)$, $\forall x \ne x_\lambda$.

(iv) $0 \le \max\left\{\max\limits_{i\in\{1,2,\ldots,p\}}\left\{\dfrac{\alpha_i\cdot(f(x)-f(x_\lambda))}{\alpha_i\cdot y_0}\right\}, \max\limits_{j\in\{1,2,\ldots,q\}}\left\{\dfrac{\beta_j\cdot(g(x)-g(x_\lambda))}{\beta_j\cdot z_0}\right\}\right\} + k_\lambda d(x_\lambda, x)$, *being the inequality strict whenever*

$$(f,g)(x) - (f,g)(x_\lambda) + k_\lambda d(x_\lambda, x)(y_0, z_0) \ne (0,0),$$

where $k_\lambda := \dfrac{\varepsilon}{\lambda}\inf\limits_{c\in C}\varphi_{D_{\bar{\rho}}, y_0}(c)$, $\alpha_i := (\bar{\rho}u + e_i)A$ for all $i \in \{1,2,\ldots,p\}$ and β_j is the jth row of B for $j \in \{1,2,\ldots,q\}$.

In the rest of the paper, we consider a nontrivial complete linear metric space (X,d). It is well known that vector variational inequality problems play an important role in vector optimization since their introduction by Giannessi in [13]. Here, we are going to study an approximate vector variational inequality problem and its relation with Henig (C,ε)-proper efficient solutions of $(\mathscr{MOP})$. For this aim, we need to use the approximate strong subdifferential stated in the next definition. It was introduced and studied by Gutiérrez, Huerga, Jiménez and Novo in [20].

From now on, $\mathscr{L}(X, \mathbb{R}^n)$ stands for the set of all continuous linear mappings from X to $\mathbb{R}^n$.

Definition 6 Let $\varepsilon \ge 0$ and $C \subset D\backslash\{0\}$ be a nonempty set. The (C,ε)-strong subdifferential of f at x_0 is defined as

$$\partial^s_{C,\varepsilon} f(x_0) := \{T \in \mathscr{L}(X,\mathbb{R}^n) : f(x) \ge_D f(x_0) - q + T(x - x_0),\ \forall q \in C(\varepsilon),\ \forall x \in X\}.$$

Also, we denote $\partial^s f(x_0) := \partial^s_{D\backslash\{0\},0} f(x_0) = \{T \in \mathscr{L}(X,\mathbb{R}^n) : f(x) \geq_D f(x_0) + T(x-x_0),\ \forall x \in X\}$.

Remark 12 If $n=1, D=[0,\infty)$ and $C=\{1\}$, then $\partial^s_{C,\varepsilon} f(x_0) = \partial_\varepsilon f(x_0)$, where $\partial_\varepsilon f(x_0)$ denotes the ε-subdifferential for scalar mappings given by Brøndsted-Rockafellar [6].

Also, if $C=\{q\}$, $q \in D\backslash\{0\}$, then the (C,ε)-strong subdifferential reduces to the ε-subdifferential for vector mappings introduced by Kutateladze [38].

Next lemma provides an easy to handle calculus rule for the (C,ε)-strong subdifferential in terms of approximate subgradients of related scalar mappings.

Given $C \subset D\backslash\{0\}$ and $\mu \in \mathbb{R}^n$ let us denote $\tau_C(\mu) := \inf_{c\in C}\langle \mu, c\rangle$.

Lemma 7 *Let $\varepsilon \geq 0$, $\emptyset \neq C \subset D\backslash\{0\}$ and $x_0 \in X$. It follows that $T \in \partial^s_{C,\varepsilon} f(x_0)$ if and only if*

$$a_i \circ T \in \partial_{\varepsilon\tau_C(a_i)}(a_i \circ f)(x_0),\ \forall i \in \{1,2,\ldots,p\}, \tag{17}$$

where a_i denotes the ith row of A, $i \in \{1,2,\ldots,p\}$.

Proof Let $T \in \partial^s_{C,\varepsilon} f(x_0)$. Then, by [20, Theorem 4.4] we know that

$$\mu \circ T \in \partial_{\varepsilon\tau_C(\mu)}(\mu \circ f)(x_0),\ \forall \mu \in D^+\backslash\{0\}.$$

It is clear that $a_i \in D^+\backslash\{0\}$ for all $i \in \{1,2,\ldots,p\}$, so in particular (17) holds. Conversely, let $\mu \in D^+\backslash\{0\}$. Then, it is easy to see that there exist $\lambda_1,\lambda_2,\ldots,\lambda_p \geq 0$ such that $\mu = \sum_{i=1}^{p} \lambda_i a_i$. Since $a_i \circ T \in \partial_{\varepsilon\tau_C(a_i)}(a_i \circ f)(x_0)$, for all $i \in \{1,2,\ldots,p\}$ it follows that

$$(a_i \circ f)(x) \geq (a_i \circ f)(x_0) - \varepsilon\tau_C(a_i) + (a_i \circ T)(x-x_0),\ \forall x \in X,\ \forall i \in \{1,2,\ldots,p\}. \tag{18}$$

Thus, taking into account (18) we have for all $x \in X$

$$\begin{aligned}
(\mu \circ f)(x) &= \sum_{i=1}^{p} \lambda_i (a_i \circ f)(x) \\
&\geq \sum_{i=1}^{p} \lambda_i (a_i \circ f)(x_0) - \varepsilon \sum_{i=1}^{p} \lambda_i \tau_C(a_i) + \sum_{i=1}^{p} \lambda_i (a_i \circ T)(x-x_0) \\
&= (\mu \circ f)(x_0) - \varepsilon \sum_{i=1}^{p} \tau_C(\lambda_i a_i) + (\mu \circ T)(x-x_0) \\
&\geq (\mu \circ f)(x_0) - \varepsilon\tau_C(\mu) + (\mu \circ T)(x-x_0).
\end{aligned}$$

Hence, by [20, Theorem 4.4] we conclude that $T \in \partial^s_{C,\varepsilon} f(x_0)$, and the proof is complete.

Remark 13 In [20, Theorem 4.6] a characterization of $\partial^s_{C,\varepsilon} f(x_0)$ was given when $D = \mathbb{R}^n_+$. Thus, Lemma 7 extends [20, Theorem 4.6] for any polyhedral ordering cone. For the convenience of the reader, note that if $D = \mathbb{R}^n_+$ then

$$\partial^s_{C,\varepsilon} f(x_0) = \prod_{i=1}^{n} \partial_{\varepsilon\tau_C(e_i)} f_i(x_0),$$

where f_i is the ith component of f and $\{e_i\}_{1\le i\le n}$ is the canonical base of $\mathbb{R}^n$.

Let $\emptyset \neq C \subset D\backslash\{0\}$, $\rho \geq 0$ and $\varepsilon_1, \varepsilon_2 \geq 0$. We define now the following approximate vector variational inequality problem:

$$\begin{array}{ll} \text{Find } x_0 \in X \text{ for which there exists } T \in \partial^s_{C,\varepsilon_1} f(x_0) \text{ satisfying} & \\ x_0 \in \mathrm{He}(F_T, X, C + D_\rho, \varepsilon_2),\ \forall x \in X, & (\mathscr{AVVIP}_{C,\varepsilon_1,\varepsilon_2,\rho}) \end{array}$$

where $F_T : X \times X \to \mathbb{R}^n$ is defined as $F_T(x,y) := T(y-x)$.

In particular, if $C = D\backslash\{0\}$ and $\varepsilon_1 = \varepsilon_2 = 0$, then we obtain the next exact vector variational inequality problem

$$\begin{array}{ll} \text{Find } x_0 \in X \text{ for which there exists } T \in \partial^s f(x_0) \text{ satisfying} & \\ x_0 \in \mathrm{He}(F_T, X, D_\rho),\ \forall x \in X. & (\mathscr{VVIP}_\rho) \end{array}$$

Remark 14 By Remark 2 it is easy to see that x_0 is a solution of $(\mathscr{AVVIP}_{C,\varepsilon_1,\varepsilon_2,0})$ if and only if there exists $\rho > 0$ such that x_0 is a solution of $(\mathscr{AVVIP}_{C,\varepsilon_1,\varepsilon_2,\rho})$. In the same way, x_0 is a solution of $(\mathscr{VVIP}_0)$ if and only if there exists $\rho > 0$ such that x_0 is a solution of $(\mathscr{VVIP}_\rho)$.

When $D = \mathbb{R}^n_+$, $\rho = 0$, $f : \mathbb{R}^s \to \mathbb{R}^n$ is a differentiable $\mathbb{R}^n_+$-convex mapping (i.e., $f(y) \geq_{\mathbb{R}^n_+} f(x) + f'(x)(y-x)$, for all $y \in X$, where $f'(x)$ denotes the Jacobian of f at x), and instead of considering proper efficient solutions in $(\mathscr{VVIP}_\rho)$ we consider efficient solutions, it follows that problem $(\mathscr{VVIP}_0)$ reduces to the well-known vector variational inequality problem studied by Yang and Goh in [51]. In this case, $\partial^s f(x) = \{f'(x)\}$, for each $x \in X$ and then T is univocally defined for each x.

Here, we are not only interested in an exact vector variational inequality problem, but also in the approximate problem $(\mathscr{AVVIP}_{C,\varepsilon_1,\varepsilon_2,\rho})$. This problem lets us derive a sufficient condition for Henig (C,ε)-proper efficient solutions of $(\mathscr{MOP})$, as it is shown in the following result.

Theorem 5 *Let $\emptyset \neq C \subset D\backslash\{0\}$, $\rho \geq 0$ and $\varepsilon, \varepsilon_1, \varepsilon_2 \geq 0$ such that $\varepsilon_1 + \varepsilon_2 = \varepsilon$. If x_0 is a solution of $(\mathscr{AVVIP}_{C,\varepsilon_1,\varepsilon_2,\rho})$, then $x_0 \in \mathrm{He}_{\mathscr{MOP}}(f, X, C + D_\rho, \varepsilon)$.*

Proof Suppose that x_0 is a solution of $(\mathscr{AVVIP}_{C,\varepsilon_1,\varepsilon_2,\rho})$. Then, there exist $T \in \partial^s_{C,\varepsilon_1} f(x_0)$ and $\bar{\rho} > 0$ such that

$$F_T(x_0, X) \cap (-C(\varepsilon_2) - \mathrm{int}\, D_{\bar{\rho}}) = \emptyset. \tag{19}$$

If $\rho > 0$ we consider without loss of generality that $\bar{\rho} = \rho$. Suppose by reasoning to the contrary that $x_0 \notin \mathrm{He}_{\mathscr{MOP}}(f, X, C + D_\rho, \varepsilon)$. Then, in particular, there exists $\bar{x} \in X$ such that

$$f(\bar{x}) - f(x_0) \in -C(\varepsilon) - \mathrm{int}\, D_{\bar{\rho}}. \tag{20}$$

Consider first that $\varepsilon_1, \varepsilon_2 > 0$. Then, $\varepsilon > 0$ and by (20) there exist $\bar{c} \in C$ and $\bar{d} \in \operatorname{int} D_{\bar{\rho}}$ such that $f(\bar{x}) - f(x_0) = -\varepsilon\bar{c} - \bar{d}$. On the other hand, since $T \in \partial^s_{C,\varepsilon_1} f(x_0)$, we have in particular that

$$f(\bar{x}) \geq_D f(x_0) - \varepsilon_1 \bar{c} + T(\bar{x} - x_0),$$

so

$$\begin{aligned} T(\bar{x} - x_0) &\in f(\bar{x}) - f(x_0) + \varepsilon_1 \bar{c} - D \\ &= -\varepsilon\bar{c} - \bar{d} + \varepsilon_1 \bar{c} - D = -\varepsilon_2 \bar{c} - \bar{d} - D \subset -C(\varepsilon_2) - \operatorname{int} D_{\bar{\rho}}, \end{aligned}$$

and we obtain a contradiction with (19), proving that $x_0 \in \mathrm{He}_{\mathscr{MOP}}(f, X, C + D_\rho, \varepsilon)$.

Now suppose that $\varepsilon_1 = \varepsilon_2 = 0$. Thus, $\varepsilon = 0$. Note that $C(0) + \operatorname{int} D_{\bar{\rho}} = \operatorname{int} D_{\bar{\rho}}$ and then by (20) there exists $\bar{d} \in \operatorname{int} D_{\bar{\rho}}$ such that $f(\bar{x}) - f(x_0) = -\bar{d}$. On the other hand, by [20, Proposition 4.3 (b)] we have $\partial^s_{C,0} f(x_0) = \partial^s f(x_0)$, so

$$f(\bar{x}) \geq_D f(x_0) + T(\bar{x} - x_0).$$

Thus,

$$T(\bar{x} - x_0) \in f(\bar{x}) - f(x_0) - D = -\bar{d} - D \subset -\operatorname{int} D_{\bar{\rho}},$$

that again contradicts (19). Cases $\varepsilon_1 > 0 = \varepsilon_2$ and $\varepsilon_2 > 0 = \varepsilon_1$ are reasoning in analogous way as above. The proof is complete.

Remark 15 For instance, for $C = q + D$, $q \in D\backslash\{0\}$, or $C = B + D$, Theorem 5 provides a sufficient condition for Henig $(C + D_\rho, \varepsilon)$-proper efficient solutions of $(\mathscr{MOP})$, for all $\rho \geq 0$.

Then, if we take $F(x, y) := f(y) - f(x)$, it is worth to remain that from Theorems 1 and 2 the $(C + D_\rho, \varepsilon)$-proper efficient solutions of $(\mathscr{MOP})$ tend to an exact Henig proper efficient solution, when $\rho > 0$, and to an exact weak efficient solution, when $\rho = 0$.

We finish the section with an EVP for solutions of $(\mathscr{AVVIP}_{C,\varepsilon_1,\varepsilon_2,\rho})$, in terms of the EVP stated in Theorem 3. The proof is straightforward from this theorem and Lemma 7.

Theorem 6 *Let $\emptyset \neq C \subset D\backslash\{0\}$ such that $C \in \mathscr{A}$, $\rho \geq 0$, $\varepsilon_1 \geq 0, \varepsilon_2 > 0$, $y_0 \in D\backslash\{0\}$ and $x_0 \in X$. If x_0 is a solution of $(\mathscr{AVVIP}_{C,\varepsilon_1,\varepsilon_2,\rho})$, then there exists $T \in \mathscr{L}(X, \mathbb{R}^n)$ such that*

$$a_i \circ T \in \partial_{\varepsilon_1 \tau_C(a_i)}(a_i \circ f)(x_0), \forall i \in \{1, 2, \ldots, p\},$$

and $\bar{\rho} > 0$ such that for each $\lambda > 0$ there exists $x_\lambda \in X$ satisfying

(i) $d(x_\lambda, x_0) \leq \lambda$;

(ii) $\displaystyle\max_{i\in\{1,2,\ldots,p\}} \left\{ \frac{(\alpha_i \cdot T)(x_\lambda - x_0)}{\alpha_i \cdot y_0} \right\} \leq -k_\lambda d(x_\lambda, x_0)$;

(iii) $\displaystyle\max_{i\in\{1,2,\ldots,p\}} \left\{ \frac{(\alpha_i \cdot T)(x_\lambda - x_0)}{\alpha_i \cdot y_0} \right\} < \max_{i\in\{1,2,\ldots,p\}} \left\{ \frac{(\alpha_i \cdot T)(x - x_0)}{\alpha_i \cdot y_0} \right\} + k_\lambda d(x, x_\lambda)$, $\forall x \neq x_\lambda$.

(iv) $0 \leq \max_{i\in\{1,2,\ldots,p\}} \left\{ \frac{(\alpha_i \cdot T)(x-x_\lambda)}{\alpha_i \cdot y_0} \right\} + k_\lambda d(x_\lambda, x)$, *for all* $x \in X$,

$0 < \max_{i\in\{1,2,\ldots,p\}} \left\{ \frac{(\alpha_i \cdot T)(x-x_\lambda)}{\alpha_i \cdot y_0} \right\} + k_\lambda d(x_\lambda, x)$, *for all* $x \in X$ *such that* $T(x - x_\lambda) + k_\lambda d(x_\lambda, x)y_0 \neq 0$,

where $k_\lambda := \frac{\varepsilon_2}{\lambda} \inf_{c\in C} \varphi_{-D_{\bar{\rho}}, y_0}(c)$ *and* $\alpha_i := (\bar{\rho}u + e_i)A$, *for all* $i \in \{1,2,\ldots,p\}$.

7 Conclusions

Given a vector equilibrium problem defined on a complete metric space and whose final space is finite dimensional, ordered by a polyhedral cone, we have obtained a variant of the EVP for a type of approximate proper solutions in the sense of Henig.

These solutions are interesting since they have a good limit behaviour when the error goes to zero. Indeed, depending on the approximation set used to define them, we have obtained sufficient conditions that guarantee that a convergent sequence of approximate proper solutions tends to an exact weak/proper/efficient solution of the vector equilibrium problem.

The aformentioned variant of the EVP is obtained through a nonlinear scalarization and its expressions depend on the matrix that defines the ordering cone, which makes them interesting computationally. Moreover, we obtain alternative EVPs for both an unconstrained and a constrained vector equilibrium problem. In the constrained case, the feasible set is given by a cone constraint and both the objective and the constraint mapping are involved in the statements of the EVP.

Finally, we have applied the obtained results to the particular case of a multiobjective optimization problem, and in this framework we have defined and studied a related vector variational inequality problem.

Acknowledgements The first and third authors are supported by Vietnam National University-Hochiminh City under the grant B2018-28-02. The work of the second and fourth authors is partially supported by Ministerio de Economía y Competitividad (Spain) under project MTM2015-68103-P (MINECO/FEDER)

References

1. Bao, T. Q., Cobzaş, S., Soubeyran, A.: Variational principles, completeness and the existence of traps in behavioral sciences. Ann Oper Res. 269, 53–79 (2018)
2. Bao, T. Q., Khanh, P. Q.: Are several recent generalization of Ekeland's variational principle more general than the original principle?. Acta Math. Vietnam 28, 345–350 (2003)
3. Bednarczuk, E. M., Przybyla, M. J.: The vector-valued variational principle in Banach spaces ordered by cones with nonempty interiors. SIAM J. Optim. 18, 907–913 (2007)
4. Benson, H. P.: An improved definition of proper efficiency for vector maximization with respect to cones. J. Math. Anal. Appl. 71, 232–241 (1979)
5. Bianchi, M., Kassay, G., Pini, R.: Ekeland's principle for vector equilibrium problems. Nonlinear Anal. 66, 1454–1464 (2007)
6. Brøndsted, A., Rockafellar, R. T.: On the subdifferentiability of convex functions. Proc. Amer. Math. Soc. 16, 605–611 (1965)
7. Chen, G. Y., Yang, X. Q.: The vector complementarity and its equivalences with the weak minimal element in ordered spaces. J. Math. Anal. Appl. 153, 136–158 (1990)

8. Del Pino, M., Felmer, P.: Semi-classical states of nonlinear Schrödinger equations: a variational reduction method. Math. Ann. 324, 1–32 (2002)
9. Ekeland, I.:, On the variational principle. J. Math. Anal. Appl. 47, 324–353 (1974)
10. El Maghri, M.: Pareto-Fenchel ε-subdifferential sum rule and ε-efficiency. Optim. Lett. 6, 763–781 (2012)
11. Geoffrion, A. M.:, Proper efficiency and the theory of vector maximization. J. Math. Anal. Appl. 22, 618–630 (1968)
12. Gerstewitz, C., Iwanow, E.: Dualität für Nichtkonvexe Vektoroptimierungsprobleme. Wissenschaftliche Zeitschrift der Technischen Hochschule Ilmenau 31, 61–81 (1985)
13. Giannessi, F.: Theorems of alternative, quadratic programs and complementarity problems. In Variational Inequalities and Complementarity Problems (R.W. Cottle, F. Giannessi, J.L. Lions eds.), John Wiley & Sons, Chichester. pp. 151–186 (1980)
14. Gong, X. H.: Efficiency and Henig efficiency for vector equilibrium problems. J. Optim. Theory Appl. 108, 139–154 (2001)
15. Gong, X. H.: Connectedness of the solution sets and scalarization for vector equilibrium problems. J. Optim. Theory Appl. 133, 151–161 (2007)
16. Gong, X. H.: Scalarization and optimality conditions for vector equilibrium problems. Nonlinear Anal. 73, 3598–3612 (2010)
17. Göpfert, A., Riahi, H., Tammer, C., Zălinescu, C.: On the vectorial Ekelands variational principle and minimal point theorems in product spaces. Nonlinear Anal. 39, 909–922 (2000)
18. Göpfert, A., Riahi, H., Tammer, C., Zălinescu, C.: Variational Methods in Partially Ordered Spaces. Springer, Berlin (2003)
19. Gutiérrez, C., Huerga, L., Jiménez, B., Novo, V.: Proper approximate solutions and ε-subdifferentials in vector optimization: Basic properties and limit behaviour. Nonlinear Anal. 79, 52–67 (2013)
20. Gutiérrez, C., Huerga, L., Jiménez, B., Novo, V.: Proper approximate solutions and ε-subdifferentials in vector optimization: Moreau-Rockafellar type theorems. J. Convex Anal. 21, 857–886 (2014)
21. Gutiérrez, C., Huerga, L., Jiménez, B., Novo, V.: Henig approximate proper efficiency and optimization problems with difference of vector mappings. J. Convex. Anal. 23, 661–690 (2016)
22. Gutiérrez, C., Huerga, L., Novo, V.: Nonlinear scalarization in multiobjective optimization with a polyhedral ordering cone. Int. Trans. Oper. Res. 45, 763–779 (2017)
23. Gutiérrez, C., Huerga, L., Novo, V., Sama, M.: Limit behaviour of approximate proper solutions in vector optimization. Submitted
24. Gutiérrez, C., Jiménez, B., Novo, V.: On approximate efficiency in multiobjective programming. Math. Meth. Oper. Res. 64, 165–185 (2006)
25. Gutiérrez, C., Jiménez, B., Novo, V.: A set-valued Ekeland's variational principle in vector optimization. SIAM J. Control. Optim. 47, 883–903 (2008)
26. Gutiérrez, C., Kassay, G., Novo, V., Ródenas-Pedregosa, J. L.: Ekeland variational principles in vector equilibrium problems. SIAM J. Optim. 27, 2405–2425 (2017)
27. Hamel, A.: Phelps' lemma, Daneš' drop theorem and Ekeland's principle in locally convex spaces. Proced. Amer. Math. Soc. 131, 3025–3028 (2003)
28. Hamel, A. H.: Equivalents to Ekelands variational principle in uniform spaces. Nonlinear Anal. 62, 913–924 (2005)
29. Henig, M. I.: Proper efficiency with respect to cones. J. Optim. Theory Appl. 36, 387–407 (1982)
30. Ioffe, A. D.: Proximal analysis and approximate subdifferentials. J. London Math. Soc. 41, 175–192 (1990)
31. Jourani, A., Thibault, L.: Verifiable conditions for openness and regularity of multivalued mappings in Banach spaces. Trans. Amer. Math. Soc. 347, 1255–1268 (1995)
32. Kaliszewski, I.: Quantitative Pareto Analysis by Cone Separation Technique. Kluwer Academic Publishers, Boston (1994)
33. Khanh, P. Q.: On Caristi-Kirk's theorem and Ekeland's variational principle for Pareto extrema. Bull. Polish Acad. Sci. Math. 37, 33–39 (1989)
34. Khanh, P. Q.: Proper solutions of vector optimization problems. J. Optim. Theory Appl. 74, 105–130 (1992)
35. Khanh, P. Q., Kruger, A., Thao, N. H.: An induction theorem and nonlinear regularity models. SIAM J. Optim. 25, 2561–2588 (2015)
36. Khanh, P. Q., Quy, D. N.: A generalized distance and Ekelands variational principle for vector functions. Nonlinear Anal. 73, 2245–2259 (2010)
37. Khanh, P. Q., Quy, D. N.: On generalized Ekeland's variational principle and equivalent formulations for set-valued mappings. J. Global Optim. 49, 381–396 (2011)

38. Kutateladze, K. K.: Convex ε-programming. Soviet Math. Dokl. 20, 391–393 (1979)
39. Liu, C. G., Ng, K. F.: Ekeland's variational principle for set-valued functions. SIAM J. Optim. 21, 41–56 (2011)
40. Loridan, P.: ε-Solutions in vector minimization problems. J. Optim. Theory Appl. 43, 265–276 (1984)
41. Luc, D. T.: Theory of Vector Optimization. Springer, Berlin 1989
42. Oettli, W., Thera, M.: Equivalents of Ekeland's principle. Bull. Aust. Math. Soc. 48, 385–392 (1993)
43. Pascoletti, A., Serafini, P.: Scalarizing vector optimization problems. J. Optim. Theory Appl. 42, 499–524 (1984)
44. Qiu, J. H.: An equilibrium version of vectorial Ekeland variational principle and its applications to equilibrium problems. Nonlinear Anal. Real World Appl. 27, 26–42 (2016)
45. Qiu, J. H.: Set-valued quasi-metrics and a general Ekelands variational principle in vector optimization. SIAM J. Control Optim. 51, 1350–1371 (2013)
46. Rockafellar, R. T.: Directionally Lipschitzian functions and subdifferential calculus. Proc. London Math. Soc. 39, 331–355 (1979)
47. Ródenas-Pedregosa, J. L.: Caracterización de Soluciones de Problemas de Equilibrio Vectoriales. Doctoral Thesis, UNED, Madrid (2018)
48. Rubinov, A.: Sublinear operators and their applications. Russian Math. Surveys 32, 115–175 (1977)
49. Tammer, C.: A generalization of Ekeland's variational principle. Optimization 25, 129–141 (1992)
50. Wu, Z., Jane, J. Y.: On error bounds for lower semicontinuous functions. Math. Program. 91, 301–314 (2002)
51. Yang, X. Q., Goh, C. J.: On vector variational inequalities: Application to vector equilibria. J. Optim. Theory Appl. 95, 431–443 (1997)