

COMPUTING THE SIGNATURES OF SUBGROUPS OF NON-EUCLIDEAN CRYSTALLOGRAPHIC GROUPS

ISMAEL CORTÁZAR AND ANTONIO F. COSTA

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Abstract. A (planar and cocompact) non-Euclidean crystallographic (NEC) group Δ is a subgroup of the group of (conformal and anti-conformal) isometries of the hyperbolic plane \mathbb{H}^2 such that \mathbb{H}^2/Δ is compact. NEC groups are classified algebraically by a symbol called signature. In this symbol there is a sign $+$ or $-$ and, in the case of sign $+$, some cycles of integers called period-cycles have an essential direction. In 1990 A.H.M. Hoare gives an algorithm to obtain the signature of a finite index subgroup of an NEC group. The process of Hoare fails in some cases in the task of computing the direction of period-cycles. In this work we complete the algorithm of Hoare, this allows us to construct a program for computing the signature of subgroups of NEC groups in all cases.

1. INTRODUCTION

A (planar) non-Euclidean crystallographic group Δ is a discrete subgroup of the group of (conformal and anti-conformal) isometries of the hyperbolic plane \mathbb{H}^2 . We shall consider only cocompact NEC groups, i. e. we assume that \mathbb{H}^2/Δ is compact. The algebraic structure of Δ is given by a symbol called signature (see [12] and [9]):

$$(g; \sigma; [m_1, \dots, m_r]; \{(n_{11}, \dots, n_{1s_1}), \dots, (n_{k1}, \dots, n_{ks_k})\})$$

where g is the genus of the surface \mathbb{H}^2/Δ , $\sigma = +$ or $-$ is the orientability character of \mathbb{H}^2/Δ , $[m_1, \dots, m_r]$ is the set of branch indices (periods) of the covering $\mathbb{H}^2 \rightarrow \mathbb{H}^2/\Delta$ with values in interior points of \mathbb{H}^2/Δ , one period for each branch value, and the ordered sets (period-cycles) of branched indices: $(n_{11}, \dots, n_{1s_1}), \dots, (n_{k1}, \dots, n_{ks_k})$, correspond to branched values in the k boundary components of \mathbb{H}^2/Δ .

Two signatures:

$$(g; \sigma; [m_1, \dots, m_r]; \{(n_{11}, \dots, n_{1s_1}), \dots, (n_{k1}, \dots, n_{ks_k})\})$$

$$(g'; \sigma'; [m'_1, \dots, m'_r]; \{(n'_{11}, \dots, n'_{1s_1}), \dots, (n'_{k1}, \dots, n'_{ks_k})\})$$

are considered equivalent if:

1. $g = g'$; $\sigma = \sigma'$;
2. $r = r'$; $(m'_1, \dots, m'_r) = (m'_{\varepsilon(1)}, \dots, m'_{\varepsilon(r)})$, $\varepsilon \in \Sigma\{1, \dots, r\}$
3. $k = k'$; $s_i = s'_{\delta(i)}$, $\delta \in \Sigma\{1, \dots, k\}$,
4. Let $\alpha = (1, \dots, s_i) \in \Sigma\{1, \dots, s_i\}$
 If $\sigma = \sigma' = +$, either:
 - a. For all i , $(n_{i1}, \dots, n_{is_i}) = (n'_{\delta(i)\theta_i(1)}, \dots, n'_{\delta(i)\theta_i(s_i)})$, $\theta_i = \alpha^{l_i}$, $0 \leq l_i \leq s_i$, or
 - b. For all i , $(n_{is_i}, \dots, n_{i1}) = (n'_{\delta(i)\theta_i(1)}, \dots, n'_{\delta(i)\theta_i(s_i)})$, $\theta_i = \alpha^{l_i}$, $0 \leq l_i \leq s_i$
 If $\sigma = \sigma' = -$, for each i , $i = 1, \dots, k$, either:
 - a. $(n_{i1}, \dots, n_{is_i}) = (n'_{\delta(i)\theta_i(1)}, \dots, n'_{\delta(i)\theta_i(s_i)})$, $\theta_i = \alpha^{l_i}$ or
 - b. $(n_{is_i}, \dots, n_{i1}) = (n'_{\delta(i)\theta_i(1)}, \dots, n'_{\delta(i)\theta_i(s_i)})$, $\theta_i = \alpha^{l_i}$

Following the terminology in [9]: in the orientable case ($\sigma = +$) corresponding pair of period-cycles are all paired in the same way, all directly or all inversely. In the non-orientable case, some are paired directly and some inversely.

Two NEC groups are isomorphic if and only if they have equivalent signatures. Each NEC admits a canonical presentation, the geometrical type of the generators and the word expressions of relations of a canonical presentation is given by the signature.

If Γ is a finite index subgroup of Δ then Γ is also an NEC group. From a presentation of Δ and the action of the generators of such presentation on the cosets Δ/Γ , the Reidemeister-Schreier method, provides a non-canonical presentation of Γ . But to obtain the signature of Γ from a non-canonical presentation is not an easy task.

If $\Delta \leq \text{Isom}^+(\mathbb{H}^2)$ then we say that Δ is a Fuchsian group and there is a direct method of Singerman [10] to obtain the signature of a subgroup Γ of Δ . If Δ is a generic NEC group this method was extended to NEC groups by Hoare in [6] (see also examples in [11]). The Hoare method uses a canonical presentation \mathcal{P} of Δ and the coset graph $\mathcal{H}(\Delta, \Gamma, \mathcal{P})$ with vertices the right cosets of Γ in Δ and edges labelled with the generators of \mathcal{P} .

The computation of the signature of subgroups of NEC groups has been carried out many times to solve different problems (see for instance [3]), one of the most popular applications is the determination of topological properties of the real part of real algebraic curves (see [4] and [2]). The consideration of algebraic curves that are not regular coverings of the Riemann sphere will be a field where Hoare method will be applied in maximal generality (see [5]).

Our first goal was to make a program code in order to automatize this process but we find some difficulties in the algorithm presented by Hoare in the case when \mathbb{H}^2/Γ is an orientable surface. In this case the period-cycles of the signature of Γ have an essential directed order, i. e. there are NEC groups with signatures differing only in the direction of the cyclic order of some of their period-cycles and that are not isomorphic (see [9]). Hence the determination of this direction is a crucial problem. The method to obtain such orders, given by Hoare in [6], fails in some special cases (see the example in Section 4). To solve this difficulty we present a new method in order to obtain the direction of the period-cycles in the signature of Γ when \mathbb{H}^2/Γ is an orientable surface (Section 5).

We have made a MATLAB program for the Hoare method with our modifications providing the signature of a subgroup Γ of an NEC group Δ in all possible cases,

starting from the image of the generators of a canonical presentation of Δ by the coset representation $\Delta \rightarrow \Delta/\Gamma$. The code and documentation about the programs are in open access in:

<http://e-spacio.uned.es/fez/collection/bibliuned:DptoMF-FCIE-Articulos>

2. FUCHSIAN AND NEC GROUPS

Let \mathbb{H}^2 be the upper half-plane model of the hyperbolic plane. The group of conformal automorphisms of \mathbb{H}^2 is $Aut^+(\mathbb{H}^2) = \{z \rightarrow \frac{az+b}{cz+d} : a, b, c, d \in \mathbb{R}, ad - bc > 0\}$. We shall denote $Aut(\mathbb{H}^2)$ the group of conformal and anticonformal automorphisms of \mathbb{H}^2 ($Aut(\mathbb{H}^2) = Aut^+(\mathbb{H}^2) \cup \{z \rightarrow \frac{a\bar{z}+b}{c\bar{z}+d} : a, b, c, d \in \mathbb{R}, ad - bc < 0\}$).

A subgroup $\Delta \subset Aut(\mathbb{H}^2)$ is called an NEC (non-Euclidean crystallographic) group when the orbit space \mathbb{H}^2/Δ is a compact surface. If $\Delta \subset Aut^+(\mathbb{H}^2)$ we say that Δ is a Fuchsian group, otherwise we call it *proper* NEC group. The orbit space \mathbb{H}^2/Δ is a compact surface that can be bordered and non-orientable.

Every NEC group has a canonical presentation of Δ with canonical generators: x_1, \dots, x_r (elliptic isometries), c_{i0}, \dots, c_{is_i} $i = 1, \dots, k$ (reflections, we shall call c_{ij}, c_{ij+1} contiguous reflections), e_1, \dots, e_k (connecting generators, elliptic or hyperbolic isometries) and either a_j, b_j $j = 1, \dots, g$ (hyperbolic isometries, in case of an orientable surface) or d_j $j = 1, \dots, g$ (glide reflections, in case of a non-orientable surface). The relations are:

- (1) $x_i^{m_i}$
- (2) $c_{is_i} e_i c_{i0} e_i^{-1}$ (connection relation)
- (3) c_{ij}^2
- (4) $(c_{ij-1}, c_{ij})^{n_{ij}}$
- (5) $e_1^{-1} \dots e_k^{-1} x_1^{-1} \dots x_r^{-1} [a_1, b_1] \dots [a_g, b_g]$ in case \mathbb{H}^2/Δ orientable (long relation)
- (6) $e_1^{-1} \dots e_k^{-1} x_1^{-1} \dots x_r^{-1} d_1^2 \dots d_g^2$ in case \mathbb{H}^2/Δ non-orientable (long relation)

(see reference [9])

Topologically \mathbb{H}^2/Δ is a genus g surface with k boundary components, that we shall say that is uniformised by Δ . Also \mathbb{H}^2/Δ has an orbifold structure with r conic points of orders m_1, \dots, m_r and s_i corner points in the i -th boundary component with orders n_{i1}, \dots, n_{is_i} .

The signature of Δ is defined by:

$$(g, \sigma, [m_1, \dots, m_r], \{(n_{11}, \dots, n_{1s_1}), \dots, (n_{k1}, \dots, n_{ks_k})\})$$

where g is the genus, the numbers m_1, \dots, m_r are called *proper periods*, the brackets $(n_{j1}, \dots, n_{js_j})$ *period-cycles* and the numbers n_{ij} are *periods*.

The hyperbolic area of the surface \mathbb{H}^2/Δ is given by:

$$S_\Delta = 2\pi(\eta g + k - 2 + \sum_{i=1}^r (1 - \frac{1}{m_i}) + \frac{1}{2} \sum_{j=1}^k \sum_{l=1}^{s_j} (1 - \frac{1}{n_{jl}}))$$

being g the genus, $\eta = 2$ for orientable surfaces, $\eta = 1$ for non-orientable ones, and k the number of connected components of the boundary.

A finite index subgroup Γ of Δ is also an NEC group and determines a $[\Delta : \Gamma]$ -fold Klein morphism $\mathbb{H}^2/\Gamma \rightarrow \mathbb{H}^2/\Delta$ ([1]), and a formula:

$$S_\Gamma = S_\Delta [\Delta : \Gamma]$$

That is known as Riemann-Hurwitz formula.

Let Δ' be another NEC group with signature:

$$(g', \sigma', [m'_1, \dots, m'_r], \{(n'_{11}, \dots, n'_{1s'_1}), \dots, (n'_{k1}, \dots, n'_{ks'_k})\})$$

then Δ' and Δ are isomorphic as abstract groups if and only if the signatures are equivalent, as defined in the introduction.

3. SKETCH OF HOARE'S ALGORITHM

For the sake of completeness we present a sketch of the Hoare algorithm in ref [6], after that we will show an example where the procedure is unable to compute the direction of period-cycles in the signature of subgroups of NEC groups.

We start from an NEC group Δ and a canonical presentation \mathcal{P} . Let $\varphi : \Delta \rightarrow \Sigma\{0, \dots, n-1\}$ be a homomorphism, of transitive image. Then we want to get the signature of the subgroup $\Gamma = \varphi^{-1}(\text{Stab}(0))$ (this homomorphism can be seen as the action of Δ on the right cosets of Γ in Δ). Equivalently, Hoare in [6], gives a way of understanding his algorithm using the Shreier graph $\mathcal{H}(\Delta, \Gamma, \mathcal{P})$ (graphical method). The graph $\mathcal{H}(\Delta, \Gamma, \mathcal{P})$ has as vertices the cosets of Δ/Γ , two vertices i, j are joined by an edge labelled with a generator a of \mathcal{P} if $\varphi(a)(i) = j$.

- (1) Let x_i an elliptic canonical generator of \mathcal{P} of order n . To each orbit of length m of $\varphi(x_i)$ corresponds a proper period of order n/m in the signature of Γ (of course if $n = m$ there is no such period). In terms of the graph $\mathcal{H}(\Delta, \Gamma, \mathcal{P})$, this proper periods correspond to cycles of length m in $\mathcal{H}(\Delta, \Gamma, \mathcal{P})$ formed by edges with label x_i .
- (2) Let c a reflection of \mathcal{P} . If $\varphi(c)(i) = i$, we shall say that c produces a reflection c_i of Γ : the c_i are the loops of $\mathcal{H}(\Delta, \Gamma, \mathcal{P})$ with label c .
- (3) If x, y are contiguous reflection generators such that xy has order n , for each orbit of the group $\langle \varphi(x), \varphi(y) \rangle$ of length m , we can have two cases:
 - (a) The orbit has two cosets (i, j) (may be $i = j$) fixed by reflection generators (one with label x and the other with label y , if m odd, or both fixed by the same element, $\varphi(x)$ or $\varphi(y)$, if m is even). In this case we say that the corresponding reflection generators of Γ are linked and produce a period cycle of order n/m . In terms of $\mathcal{H}(\Delta, \Gamma, \mathcal{P})$ this step is directly interpreted using paths labelled alternatively with two contiguous reflections and joining two reflection loops
 - (b) The orbit has not fixed classes for reflection generators of \mathcal{P} , then it yields an elliptic generator of order $2n/m$. In this case we consider bicoloured cycles (with two reflection labels) in $\mathcal{H}(\Delta, \Gamma, \mathcal{P})$.
- (4) Now the cycles of period cycles of the signature of Γ are given by the periods obtained in 3.a in cycles of linked reflections of Γ . This process give us an order (up direction) for period-cycles.
- (5) In this case the approach using $\mathcal{H}(\Delta, \Gamma, \mathcal{P})$ is more natural: the sign in the signature of Γ is $+$ if and only if the vertices of $\mathcal{H}(\Delta, \Gamma, \mathcal{P})$ can be bicoloured, in such a way that edges labelled with conformal generators of \mathcal{P} join vertices with the same colour and the edges labelled with anticonformal generators join vertices with two different colours (except reflection loops) (see [8])
- (6) Using Riemann-Hurwitz relation we can compute the genus of \mathbb{H}^2/Γ .
- (7) Assume the sign of the signature of Γ is $+$. In this case Hoare gives two methods to obtain the direction of period-cycles. The first one needs to construct a complete presentation for Γ (of a special form see [7]) and adjust the directions of period-cycles to satisfy some relations, but there is no an algorithm to do that and we shall present an example where this method

does not give an answer (we call this process the long/connection relations method). For the second method, using $\mathcal{H}(\Delta, \Gamma, \mathcal{P})$, the words of Hoare are: “we determine the directions of the period-cycles of Γ by a process which traverses each edge of $\mathcal{H}(\Delta, \Gamma, \mathcal{P})$ twice, once in each direction”. The problem is that there are cycle-periods obtained from cycles of reflection loops in $\mathcal{H}(\Delta, \Gamma, \mathcal{P})$ with only one vertex and no edges of $\mathcal{H}(\Delta, \Gamma, \mathcal{P})$ (see how works this procedure in Example 1 of [6], where the use of paths with more than a vertex is essential). Hoare’s method to obtain the direction of period-cycles of Γ works in many cases, but when Δ has more than one period cycle and in relatively simple cases his method fails.

4. EXAMPLE WHERE DIRECTION OF PERIOD-CYCLES CANNOT BE FOUND BY HOARE METHOD

Let Δ be a group with signature

$$(0; +; [2, 2]; \{(6, 12, 24), (6, 12, 24)\})$$

and canonical presentation:

- (1) Generators:
 - x_1, x_2 elliptic of order 2
 - $c_{10}, c_{11}, c_{12}, c_{13}$ reflections of first period cycle
 - $c_{20}, c_{21}, c_{22}, c_{23}$ reflections of second period cycle
 - e_1, e_2 connection generators
- (2) Relations:
 - x_1^2, x_2^2, c_{ij}^2
 - $(c_{10}c_{11})^6, (c_{11}c_{12})^{12}, (c_{12}c_{13})^{24}, c_{13}e_1c_{10}e_1^{-1}$
 - $(c_{20}c_{21})^6, (c_{21}c_{22})^{12}, (c_{22}c_{23})^{24}, c_{23}e_2c_{20}e_2^{-1}$
 - $e_1^{-1}e_2^{-1}x_1^{-1}x_2^{-1}$

We consider the homomorphism with transitive image: $\varphi : \Delta \rightarrow \Sigma\{0, 1, 2, 3\}$ defined by:

$$\begin{array}{llll} x_1 \rightarrow (0, 1) & x_2 \rightarrow (0, 1) & & \\ c_{10} \rightarrow (0, 3) & c_{11} \rightarrow (0, 2) & c_{12} \rightarrow (0, 2) & c_{13} \rightarrow (0, 3) \\ c_{20} \rightarrow (0, 3) & c_{21} \rightarrow (0, 3) & c_{22} \rightarrow (0, 2) & c_{23} \rightarrow (0, 3) \\ e_1 \rightarrow id & e_2 \rightarrow id & & \end{array}$$

We want to compute the signature of the subgroup $\Gamma = \varphi^{-1}(\text{Stab}(0))$.

Using Hoare algorithm we have that Γ has:

- (1) Genus 2
- (2) The sign of signature Γ is + (bipartite sets $\{0, 1\}\{2, 3\}$)
- (3) Proper periods: 2, 2, 2, 12, 6
- (4) Four period-cycles (each period cycle of Δ generates two) with periods:
 - (a) 6, 12, 24
 - (b) 2, 12, 8
 - (c) 6, 12, 24
 - (d) 8, 4, 6

But both connection generators have associated the identical permutation, so it is not possible to use the long/connection relations to obtain the relative direction of period cycles. The second method of Hoare does not work since there are paths of linked reflection loops with only one vertex:

$$c_{10_1}, c_{11_1}, c_{12_1}, c_{13_1} \text{ and } c_{20_1}, c_{21_1}, c_{22_1}, c_{23_1}.$$

There is no edges to be traversed twice in the paths defining these period-cycles.

5. A NEW ALGORITHM TO GET THE DIRECTION OF PERIOD-CYCLES

Let Δ an NEC group with signature:

$$(g, \pm, [m_1, \dots, m_r], \{(n_{11}, \dots, n_{1s_1}), \dots, (n_{k1}, \dots, n_{ks_k})\})$$

We want to compute the signature of $\Gamma \subset \Delta$ with $[\Delta : \Gamma] = n$. There is a representation $\varphi : \Delta \rightarrow \Sigma\{0, 1, \dots, n-1\}$ given by the action of Δ on the right cosets of Δ/Γ .

We consider a canonical NEC presentation \mathcal{P} of Δ . Assume that the reflections generators of \mathcal{P} are:

$$c_{10}, \dots, c_{1s_1}, \dots, c_{i0}, \dots, c_{is_i}, \dots, c_{k0}, \dots, c_{ks_k}$$

being $(c_{ij}c_{ij+1})^{n_{ij+1}} = 1$

Note that the presentation \mathcal{P} gives an order to the period-cycles in the signature of Γ .

Assume that Γ has sign $+$ in its signature. There is a bipartition of the vertices in the coset graph $\mathcal{H}(\Delta, \Gamma, \mathcal{P})$ as described in [6], so we can assign a ‘‘colour’’ (black or white) to each vertex.

We start on a vertex t of $\mathcal{H}(\Delta, \Gamma, \mathcal{P})$ where there is reflection generator c_{ij} of Δ such that $\varphi(c_{ij})(t) = t$, i.e. the edge in $\mathcal{H}(\Delta, \Gamma, \mathcal{P})$ starting in t and with label c_{ij} is a loop. It gives place to a reflection generator c_{ij-t} of Γ .

If t is *white* then we consider the sequence:

$$t_1 = \varphi(c_{ij+1})(t), t_2 = \varphi(c_{ij})(t_1), t_3 = \varphi(c_{ij+1})(t_2), t_3 = \varphi(c_{ij})(t_3), \dots$$

Until $t_p = t_{p+1}$ (it is possible $t = t_1$).

If p is even then the colour of t_p is also white (as we pass from one to the other multiplying by an even number of reflections). One part of the sequence of reflections of a period cycle of Γ will be c_{ij-t} , c_{ij+1-t_p} and their product has order $\frac{n_{ij+1}}{p+1}$. Then we have a period $\frac{n_{ij+1}}{p+1}$ in the period-cycle that we are constructing in the signature of Γ . We restart the procedure with the generator c_{ij+1} that fixes the white vertex t_p i.e. $\omega(c_{ij+1})(t_p) = t_p$, in this way we shall obtain the next period in the period-cycle in an allowed direction for the signature of Γ .

If p is odd then the colour of t_p is black (as we pass from one to other multiplying an odd number of reflections). We obtain a period $\frac{n_{ij+1}}{p+1}$ and we restart the procedure (for a black vertex) with the generator c_{ij} that besides t fixes also the vertex t_p . The process applied to c_{ij} and t_p will provide the next period in the direction of the period-cycle.

Now if t is *black* we have:

$$t_1 = \varphi(c_{ij-1})(t), t_2 = \varphi(c_{ij})(t_1), t_3 = \varphi(c_{ij-1})(t_2), t_3 = \varphi(c_{ij})(t_3), \dots$$

Until $t_p = t_{p+1}$ (it is possible $t = t_1$).

If p is even then the colour of t_p is also black. We have a period $\frac{n_{ij+1}}{p+1}$ and we restart the procedure with the generator c_{ij-1} that fixes the vertex t_p i.e. $\varphi(c_{ij-1})(t_p) = t_p$.

If p is odd then the colour of t_p is white (as we pass from one to the other multiplying an odd number of reflections) and we have a period $\frac{n_{ij+1}}{p+1}$. We restart the procedure (for a white vertex) with the generator c_{ij} that besides t fixes also the vertex t_p .

Equivalently: If we are in a white vertex we look at the orbits of the action of the dihedral group $\langle \varphi(c_{ij}), \varphi(c_{ij+1}) \rangle$ there must be one that contains t and another vertex t_p fixed by c_{ij+1} (if the length of the orbit is odd) or by c_{ij} if the length is even. If we are in a black vertex we look at the orbits of the action of the dihedral group $\langle \varphi(c_{ij-1}), \varphi(c_{ij}) \rangle$ and also there must be one that contains t and another vertex t_p fixed by c_{ij-1} (if the length of the orbit is odd) or by c_{ij} if the length is even.

After a finite number of steps we come back to the vertex t and the generator c_{ij} and we have an oriented cycle. We continue the process with the loops that have not yet used to obtain all the directed cycles of the signature of Γ .

Note that we have two situations when we must do a “change of direction”, passing from orbits of $\langle \varphi(c_{ij}), \varphi(c_{ij+1}) \rangle$ to orbits of $\langle \varphi(c_{ij-1}), \varphi(c_{ij}) \rangle$ or viceversa:

- When the colour of the vertex change.
- When the length of the previous orbit is even.

Proof.

The canonical presentation \mathcal{P} of Δ is associated to a canonical fundamental polygon \mathcal{F} in \mathbb{H}^2 (see [9]). The region \mathcal{F} has an orientation that produces an orientation in $\partial\mathcal{F}$ and an order to the sides of \mathcal{F} . The fixed point line of the reflection generator c_{ij} of \mathcal{P} contains a side of \mathcal{F} , then the order of the sides of \mathcal{F} gives an order to the reflection generators and also to the period cycles in the signature of Δ . This order on the reflection generators is expressed by the order c_{ij}, c_{ij+1} .

If $\{\Gamma, g_1, \dots, \Gamma g_{n-1}\}$ is a set of right cosets representatives of Δ/Γ then:

$$\mathcal{F}_\Gamma = \mathcal{F} \cup g_1(\mathcal{F}) \cup \dots \cup g_n(\mathcal{F})$$

is a fundamental polygon of \mathcal{F}_Γ .

The orientation of \mathcal{F}_Γ given by the orientation of \mathcal{F} , produces an orientation on \mathbb{H}^2/Γ . Such orientation gives an orientation to the components of $\partial\mathbb{H}^2/\Gamma$ that provides the direction of period-cycles of Γ that we are looking for. Hence the direction of period-cycles is given by orientation of $\partial\mathcal{F}_\Gamma$.

We give colour white to the polygons \mathcal{F} and $g_i\mathcal{F}$, where g_i is orientation preserving and black to the other ones, this coloration does not depends on the representatives g_i since there is a sign $+$ in the signature of Γ and the characterization in 5 of Section 3 (see [8])

In the white regions the order on the sides given by the orientation of \mathcal{F}_Γ will be the same that the order on the sides of $\partial\mathcal{F}$ but the contrary in the black regions.

Let c_{ij} be a canonical reflection generator of Δ such that $\varphi(c_{ij-1})(v) = v$ and c_{ij_v} be the corresponding reflection generator of Γ . If v corresponds to a white polygon, the following reflection generator to c_{ij_v} will be either c_{ij+1_w} or $c_{ij_w'}$. The colour of w or w' is given by the orientation preserving or reversing character of the transformation in $\langle c_{ij}, c_{ij+1} \rangle$ sending the polygon corresponding to v to the polygon with label w or w' . Finally if v corresponds to a region black the following reflection generator to c_{ij_v} is c_{ij-1_w} or $c_{ij_w'}$. This justify the method above. \square

Example.

Using this method to the example in Section 4 we have that the signature of Γ is:

$$(2; +; [2, 2, 2, 2, 12, 6]; \{(6, 12, 24), (8, 12, 2), (6, 12, 24), (8, 4, 6)\})$$

We resume the computation below:

Bicolouration: White vertices: 0, 1, black vertices: 2, 3.

Starting reflection generator: c_{10_1} . Path of linked reflections (path with only one white vertex): $c_{10_1}, c_{11_1}, c_{12_1}, c_{13_1}$.

- Directed period-cycle: (6, 12, 24).

Starting reflection generator: c_{10_2} (black vertex, backward direction)

Orbit containing 2 of $\langle \varphi(c_{10}) = \varphi(c_{13}), \varphi(c_{12}) \rangle$: $\{0, 2, 3\}$. The following reflection generator is c_{12_3} (black vertex), period: $24/3 = 8$

Orbit containing 3 of $\langle \varphi(c_{12}), \varphi(c_{11}) \rangle$: $\{3\}$. The following reflection is c_{11_3} (black vertex), period: 12

Orbit containing 3 of $\langle \varphi(c_{11}), \varphi(c_{10}) \rangle$: $\{0, 2, 3\}$. The following reflection is c_{10_2} (black vertex), period: $6/3 = 2$.

- Directed period-cycle (8, 12, 2)

Starting reflection generator c_{10_1} . Path of linked reflections (in a white vertex): $c_{20_1}, c_{21_1}, c_{22_1}, c_{23_1}$.

- Directed period-cycle: (6, 12, 24).

and $c_{20_1}, c_{21_1}, c_{22_1}, c_{23_1}$.

The reflections in the last period are: c_{20_2} (conjugate to c_{23_2}), c_{22_3}, c_{21_2} .

- The last directed period-cycle is (8, 4, 6)

6. A MORE COMPLEX EXAMPLE

The new algorithm quickly gives the signature of rather complex cases as the example that follows, that could not be solved using the original Hoare algorithm:

Let Δ a group with signature:

$$(2; -; [4, 4]; \{(24, 6, 12), (6, 12, 24)\})$$

with the generators of a canonical presentation:

$$\{a_1, a_2, x_2, x_2, e_1, e_2, e_3, c_{10}, c_{11}, c_{12}, c_{13}, c_{20}, c_{21}, c_{22}, c_{23}\}$$

And $\varphi : \Delta \rightarrow \Sigma\{0, 1, \dots, 7\}$ defined by:

$$\begin{array}{llll} a_1 \rightarrow (0, 1)(2, 5)(3, 4)(6, 7) & a_2 \rightarrow (0, 1)(2, 3)(4, 5)(6, 7) & & \\ x_1 \rightarrow (0, 2, 4, 6)(61, 7) & x_2 \rightarrow (0, 4) & & \\ c_{10} \rightarrow (1, 2) & c_{11} \rightarrow (0, 1)(3, 4) & c_{12} \rightarrow (0, 1)(2, 3)(6, 7) & c_{13} \rightarrow (0, 3) \\ e_1 \rightarrow (0, 2, 4)(1, 3) & & & \\ c_{20} \rightarrow (0, 3)(4, 7) & c_{21} \rightarrow (1, 2) & c_{22} \rightarrow (1, 4) & c_{23} \rightarrow (1, 6)(0, 3) \\ e_2 \rightarrow (0, 4, 6)(1, 3, 7) & & & \end{array}$$

We want to compute $\Gamma = \varphi^{-1}(\text{Stab}(0))$.

We summarize our calculations below:

Vertices fixed by reflections (giving reflection generators):

$$\begin{array}{l} c_{10} : \{0, 3, 4, 5, 6, 7\}, c_{11} : \{2, 5, 6, 7\}, c_{12} : \{4, 5\}, c_{13} : \{1, 2, 4, 5, 6, 7\}, c_{20} : \{1, 2, 5, 6\}, c_{21} : \\ \{0, 3, 4, 5, 6, 7\}, c_{22} : \{0, 2, 3, 5, 6, 7\}, c_{23} : \{2, 4, 5, 7\}, c_{30} : \{1, 2, 5, 6\}, c_{31} : \{0, 3, 4, 5, 6, 7\}, \\ c_{32} : \{0, 2, 3, 5, 6, 7\}, c_{33} : \{2, 4, 5, 7\} \end{array}$$

Orbits of dihedral groups:

$$\varphi \langle c_{10}, c_{11} \rangle = \{0, 1, 2\}\{3, 4\}\{5\}\{6\}\{7\}, \varphi \langle c_{11}, c_{12} \rangle = \{0, 1\}\{2, 3, 4\}\{5\}\{6, 7\}, \varphi \langle c_{12}, c_{13} \rangle = \{0, 1, 2, 3\}\{4\}\{5\}\{6, 7\}$$

Equivalence from c_{13_i} to c_{10_j} , $i \rightarrow j : 1 \rightarrow 3, 2 \rightarrow 4, 4 \rightarrow 0, 5 \rightarrow 5, 6 \rightarrow 6, 7 \rightarrow 7$

Second period-cycle:

$$\varphi \langle c_{20}, c_{21} \rangle = \{0, 3\}\{1, 2\}\{5\}\{6\}\{4, 7\}, \varphi \langle c_{21}, c_{22} \rangle = \{0\}\{1, 2, 4\}\{3\}\{5\}\{6\}\{7\}, \varphi \langle c_{22}, c_{23} \rangle = \{0, 3\}\{1, 4, 6\}\{2\}\{5\}\{7\}$$

Equivalence from c_{23_i} to c_{20_j} , $i \rightarrow j : 2 \rightarrow 2, 4 \rightarrow 6, 5 \rightarrow 5, 7 \rightarrow 1$

Bipartition of vertices: $\{0, 2, 4, 6\}$ white, $\{1, 3, 5, 7\}$ black.

Applying the Hoare method with the new algorithm for direction of period-cycles we get (w,b indicates the colour of the vertex and the direction of the scan w(forward), b(backward)):

$c_{10_0}(w), c_{11_2}(w), c_{12_4}(w), c_{13_4}(w)$ dihedral orders (8, 2, 12, 1).

$c_{10_3}(b), c_{13_1}(b), c_{13_2}(w), c_{10_4}(w)$ dihedral orders (1, 3, 1, 12).

$c_{10_5}(b), c_{13_5}(b), c_{12_5}(b), c_{11_5}(b)$ dihedral orders (1, 12, 6, 24).

$c_{10_6}(w), c_{11_6}(w), c_{11_7}(b), c_{10_7}(b), c_{13_7}(b), c_{13_6}(w)$ dihedral orders (24, 3, 24, 1, 6, 1).

$c_{20_1}(b), c_{23_7}(b), c_{22_7}(b), c_{21_7}(b), c_{21_4}(w), c_{22_2}(w), c_{23_2}(w), c_{20_2}(w)$ dihedral orders (1, 24, 12, 3, 4, 24, 1, 3).

$c_{20_5}(b), c_{23_5}(b), c_{22_5}(b), c_{21_5}(b)$ dihedral orders (1, 24, 12, 6).

$c_{20_6}(w), c_{21_6}(w), c_{22_6}(w), c_{23_4}(w)$ dihedral orders (6, 12, 8, 1).

$c_{21_0}(w), c_{22_0}(w), c_{22_3}(b), c_{21_3}(b)$ dihedral orders (12, 12, 12, 3).

.

So the signature of Γ is:

$$(16; +; [2, 4, 4, 2, 4, 4, 4, 4, 4, 6]; \{(8, 12, 2), (3, 12), (12, 6, 26), (24, 3, 24, 6), (24, 12, 3, 4, 24, 3), (24, 12, 6), (6, 12, 8), (12, 12, 12, 3)\})$$

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DEPARTAMENTO DE MATEMÁTICAS FUNDAMENTALES, FACULTAD DE CIENCIAS, UNED, 28040 MADRID
SPAIN

E-mail address: `icortazar3@gmail.com`

DEPARTAMENTO DE MATEMÁTICAS FUNDAMENTALES, FACULTAD DE CIENCIAS, UNED, 28040 MADRID
SPAIN

E-mail address: `acosta@mat.uned.es`