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Material geometry

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1 Introduction

The intimate kinship between Continuum Mechanics and Differential Geometry is apparent from the very fact that a continuum is, by definition, a differentiable manifold. Moreover, just as in classical Analytical Mechanics the configuration space of a system of particles is regarded as a finite-dimensional differentiable manifold, so too the configuration space of a continuum can be regarded as an infinite dimensional manifold of maps [6, 15]. Thus, the kinematic aspects of the discipline are completely embraceable within a differential geometric context. One of the seminal contributions of Walter Noll [14] was the realization that the specification of a constitutive response of a materially uniform body automatically endows the body with an additional differential geometric structure that we call *material geometry*. This point of view is consonant with the rich tradition of the theory of continuous distribution of defects, pioneered earlier by Kondo [10], Kroener [11], Eshelby [7], Bilby [1], Frank [8], and others. Noll's original work was soon significantly extended by Wang [16] and Bloom [2]. Furthermore, the formalism of G-structures [3] and groupoids [4] was shown to be the natural tool to express the geometry associated with the constitutive structure of uniform bodies of simple and structured media.

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Our objective in this article is to present a unified treatment of material geometry from the vantage point of the geometric theory of groupoids. What is gained in so doing is the awareness that to any constitutive response of a body, whether uniform or not, one can uniquely associate a corresponding *material groupoid*. If the body happens to be smoothly uniform, an associated *material Lie algebroid* becomes available and plays a fundamental role in the determination of homogeneity conditions. If, on the other hand, the body is not uniform, we show that, under certain conditions of smoothness, a unique singular *material distribution* can be defined that divides the body into parts of various dimensions, each of which is smoothly uniform. Some of these results have been separately introduced in recent publications [].

2 Constitutive response and material isomorphisms

The simplest example of a geometric material structure is provided by (first-grade) elastic materials. The *constitutive response* at a point X of a *body manifold* $\mathcal{B}$ is defined as follows. Let $\mathcal{L}(U, V)$ denote the space of linear isomorphisms between two vector spaces U, V of the same finite dimension. A constitutive response at $X \in \mathcal{B}$ is a (smooth) map

$$\psi_X : \mathcal{L}(T_X\mathcal{B}, \mathbb{R}^3) \to \mathcal{R}, \tag{1}$$

where $T_X\mathcal{B}$ is the tangent space of $\mathcal{B}$ at X and where $\mathcal{R}$ is the space of values. The *Cauchy stress*, for example, takes values in the space of contravariant symmetric tensors on $\mathbb{R}^3$. The constitutive response of the body is the collection of the constitutive responses of its points. A *configuration* κ of the body is an embedding

$$\kappa : \mathcal{B} \to \mathbb{E}^3, \tag{2}$$

where $\mathbb{E}^3$ is the standard Euclidean space, whose tangent space at each point is precisely $\mathbb{R}^3$. For any configuration κ we can evaluate the field ψ (such as the Cauchy stress field) in that configuration as the map $\psi_\kappa : \mathcal{B} \to \mathcal{R}$ given by

$$\psi_\kappa(X) = \psi_X(\kappa_*(X)), \tag{3}$$

where $\kappa_*(X)$ is the derivative of κ at X. Notice that $\kappa_*(X)$ belongs indeed to $\mathcal{L}(T_X\mathcal{B}, \mathbb{R}^3)$, as implied by Equation (2) and the notion of derivative of a map between manifolds.

These definitions do not make use of any specific reference configuration. If such a reference configuration, say κ_0, is specified, then the constitutive response with respect to that configuration can be deduced by composition of maps. The use of a reference configuration is justified in practical applications by the need to have an explicit analytic expression for the constitutive response, for which the terminology *constitutive equation* is usually applied. Notice that the 'master' constitutive response ψ defined on the body manifold guarantees the usual consistency rules between the subsidiary constitutive equations referred to different reference configurations.

Two body points, $X, Y \in \mathcal{B}$ are *materially isomorphic* if there exists a map $\mathbf{P} \in \mathcal{L}(T_X\mathcal{B}, T_Y\mathcal{B})$ such that

$$\psi_Y(\mathbf{F}) = \psi_X(\mathbf{FP}) \qquad \forall \mathbf{F} \in \mathcal{L}(T_Y\mathcal{B}, \mathbb{R}^3). \tag{4}$$

The physical meaning of the existence of a material isomorphism $\mathbf{P}$ between X and Y is that these two points are made of the same material. A body all of whose points are mutually materially isomorphic is said to be *materially uniform.* A body is *smoothly uniform* if it can be covered with materially uniform open sets such that in each of them the material isomorphisms can be chosen smoothly.

A material automorphism at a point $X \in \mathcal{B}$ is nothing but a *material symmetry* of X. Under a material symmetry, the constitutive response remains *invariant*, as implied by Equation (4) when setting $Y = X$. The collection of all material symmetries at X forms a group $\mathcal{G}_X$, known as the *symmetry group* at X. A group is essentially an algebraic structure. If it happens to be a *Lie group*, $\mathcal{G}_X$ also has the differential geometric structure of a smooth manifold. Thus, every point of a material body endowed with a constitutive response carries its own algebraic and/or geometric structure. In a celebrated imagery [17], we may think of the body as a bathroom floor made of infinitesimal tiles, each of which has some symmetries (rotations and reflections). These are *local symmetries.* On the other hand, different tiles may also be compared with each other. If two tiles are congruent, they can be mutually transplanted so as to occupy each other's former place. Under this action, the floor remains invariant. It seems justifiable to call these congruences *distant symmetries* and to ask the question as to whether there might exist an algebraic/geometric structure that can encompass *both local and distant symmetries* under a single umbrella enjoying properties similar, if not identical, to those of a group.

3 The geometry induced by the material isomorphisms

3.1 An intuitive view

Each material isomorphism $\mathbf{P} : T_X\mathcal{B} \to T_Y\mathcal{B}$ can be imagined as an arrow with its tail at X and its tip at Y. Let $\mathcal{Z}$ denote the set of all these arrows corresponding to a given constitutive response. This set is never empty. Indeed, it contains at the very least the identity automorphisms $id_X : T_X\mathcal{B} \to T_X\mathcal{B}$ for all $X \in \mathcal{B}$. In the arrow description, these are loop-shaped arrows, one at each body point. Moreover, if a point X has additional non-trivial material symmetries, we obtain a collection of loop-shaped arrows at X which embody the local symmetry group $\mathcal{G}_X$.

Consider two different points X, Y that happen to be materially isomorphic. In that case, we include an arrow $\mathbf{P}(X, Y)$ from X to Y. Since material isomorphism is an equivalence relation, we must also include the reverse arrow $\mathbf{P}(Y, X) = \mathbf{P}^{-1}(X, Y)$. Moreover, if $\mathbf{P}(Y, Z)$ is an additional material isomorphism between Y and Z, by transitivity we must also include the composite arrow $\mathbf{P}(X, Z) = \mathbf{P}(Y, Z)\, \mathbf{P}(X, Y)$, as suggested in Figure 1. In other words, arrows are composed in the standard tip-to-tail fashion, and in no other way. Two arrows cannot be composed unless the tip of the first coincides with the tail of the second. We will call the collection $\mathcal{P}$ of all these arrows the *material groupoid* associated with the given constitutive response, a terminology that will be presently justified. At this stage we can state that

> *To any given material response $\psi_X(\cdot)$, whether or not continuous with respect to X, we can associate a uniquely defined material groupoid.*

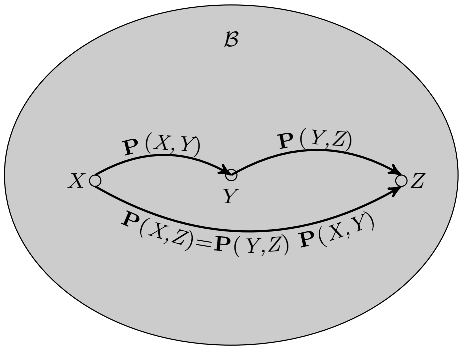

Fig. 1 Material isomorphisms as groupoid arrows and their composition

3.2 **Groupoids: definition and terminology**

Following [12], a *groupoid* $\mathcal{Z} \rightrightarrows \mathcal{M}$ consists of

- A *total set* $\mathcal{Z}$, whose elements are called *arrows*;
- A *base set* $\mathcal{M}$, whose elements are called *objects*;
- Two surjective projection maps, $\alpha : \mathcal{Z} \to \mathcal{M}$ and $\beta : \mathcal{Z} \to \mathcal{M}$, called the *source map* and the *target map*, respectively;
- An injective map $i : \mathcal{M} \to \mathcal{Z}$, called the *object inclusion map*;
- An associative binary internal operation in $\mathcal{Z}$ called *composition* or *product* defined for, and only for, all pairs $u, v \in \mathcal{Z}$ such that $\beta(u) = \alpha(v)$. The result of the composition is denoted as vu.

The product, moreover, satisfies

- $\alpha(vu) = \alpha(u)$ and $\beta(vu) = \beta(v)$;
- $\alpha \circ i(X) = \beta \circ i(X) = X\ \forall X \in \mathcal{M}$;
- $u(i \circ \alpha(u)) = u$ and $(i \circ \beta(u))u = u\ \ \forall u \in \mathcal{Z}$;
- For each $u \in \mathcal{Z}$ there is an *inverse* $u^{-1} \in \mathcal{Z}$ such that $\alpha(u^{-1}) = \beta(u)$, $\beta(u^{-1}) = \alpha(u)$ and $u^{-1}u = i \circ \alpha(u)$, $uu^{-1} = i \circ \beta(u)$.

Although the overly formal statement of these properties is somewhat intimidating, they simply state that the source and target of a product are provided by the source of one factor and the target of the other, and that the object inclusion map acts as an assignation of a *unit element* (identity arrow) to each point of the base set $\mathcal{M}$. Finally, for each arrow u with source at $X = \alpha(u)$ and target at $Y = \beta(u)$, there is an opposite arrow u^{-1} with source at Y and target at X. Their compositions, uu^{-1} and $u^{-1}u$, produce the respective units at the target and the source of u. It is not difficult to prove that the units and inverses are unique.

For each object $X \in \mathcal{M}$, the subsets $\mathcal{Z}_X = \alpha^{-1}(X) \subset \mathcal{Z}$ and $\mathcal{Z}^X = \beta^{-1}(X) \subset \mathcal{Z}$ are called the *α-fibre* and the *β-fibre* over X, respectively. Accordingly, the subset $\mathcal{Z}_X^Y = \mathcal{Z}_X \cap \mathcal{Z}^Y$ is the collection of all arrows that start at X and end at Y. Thus, $\mathcal{Z}_X^X$ is the collection of all loop-shaped arrows at$X$. It is not difficult to conclude, from the rules for the product, that each of these subsets $\mathcal{Z}_X^X$ is a group, known as the *vertex group* at X.

A groupoid is *transitive* if $\mathcal{Z}_X^Y \neq \emptyset$ for all pairs $X, Y \in \mathcal{M}$. In a transitive groupoid all the vertex groups are conjugate of each other in the sense that

$$\mathcal{Z}_Y^Y = u\, \mathcal{Z}_X^X\, u^{-1}, \tag{5}$$

where $u \in \mathcal{Z}_X^Y$. Any one of these conjugate groups can be declared to be the *structure group* of the transitive groupoid.

A groupoid is *totally intransitive* if, for all $X, Y \in \mathcal{M}$, it happens that $X \neq Y \Rightarrow \mathcal{Z}_X^Y = \emptyset$. Put differently, no two distinct objects are connected by an arrow.

We can always define an equivalence relation $\sim$ on $\mathcal{M}$ by stipulating that $X \sim Y$ if, and only if, $\mathcal{Z}_X^Y \neq \emptyset$. The corresponding equivalence classes are denoted by $\mathcal{M}_X$. The restriction $\mathcal{Z}_X$ of $\mathcal{Z}$ to an equivalence class $\mathcal{M}_X$ is obviously a transitive groupoid, called the *transitivity component* of $\mathcal{Z}$ that contains X. In short, every groupoid can be seen as the disjoint union of its transitivity components. In particular, a totally intransitive groupoid is precisely the disjoint union of its vertex groups. Some of these concepts are illustrated, albeit naively, in Figure 2.

Just as in the case of a group, a groupoid is defined as a purely algebraic structure. Topological and geometrical aspects are introduced whenever the sets of arrows and objects, the maps relating them and the composition and inversion operations enjoy further topological and differential properties. Thus, a *topological groupoid* $\mathcal{Z} \rightrightarrows \mathcal{M}$ is a groupoid for which $\mathcal{Z}$ and $\mathcal{M}$ are topological spaces and the projections α, β, the inclusion map i and the operations of composition and inversion are continuous in the respective topologies. A *Lie groupoid* $\mathcal{Z} \rightrightarrows \mathcal{M}$ is a groupoid where $\mathcal{Z}$ and $\mathcal{M}$ are smooth manifolds, the projections α, β are surjective submersions, the object inclusion map i is smooth and so are the operations of composition and inversion.[1]

[1] The assumption of smoothness of the inversion is not necessary, since it is implied by the other properties. Mackenzie [12] distinguishes between Lie groupoids and differentiable groupoids, the latter not being necessarily locally trivial.

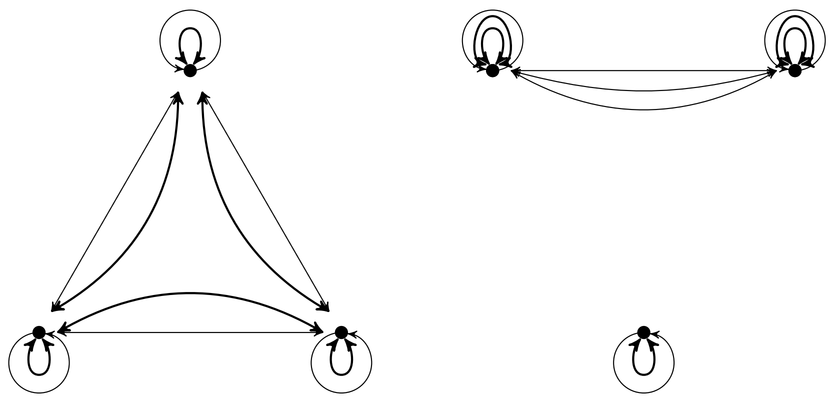

Fig. 2 A groupoid on a discrete base set. The transitivity components are clearly visible.

3.3 Correspondences with Continuum Mechanics

Combining the intuitive considerations developed in Section 3.1 with the formal definition of a groupoid introduced in Section 3.2, we have at our disposal the material groupoid $\mathcal{P} \rightrightarrows \mathcal{B}$ associated with the constitutive response $\psi_X(\cdot)$, and we can assign to the maps α, β and i and to the operations of composition and inversion the obvious physical meanings implied in the very definition of material isomorphisms. The vertex group $\mathcal{P}_X^X$ is precisely the material symmetry group $\mathcal{G}_X$.

Comparing the concept of a materially uniform body with the definition of a transitive groupoid we conclude that

A body $\mathcal{B}$ is materially uniform if, and only if, the material groupoid $\mathcal{P}$ is transitive.

At the other extreme, a totally intransitive material groupoid describes a situation in which no two material points of the body are made of the same material. Between these two extreme cases, we can state that

A constitutive response partitions the body into uniform components, each one corresponding to a transitivity component of the material groupoid.

Assume that the constitutive response $\psi_X(\cdot)$ is a smooth function of its arguments and that the body $\mathcal{B}$ is considered as a smooth manifold. It may appear that these properties imply that the resulting material groupoid is a Lie groupoid, but this is not the case in general, as can be shown by counterexamples. Even when the body is smoothly uniform, so that the material groupoid is transitive, a proof that smoothness of the constitutive law implies that the material groupoid is a Lie groupoid is not available, although counterexamples have not been found.

3.4 **Subgroupoids**

A map between two groupoids, $\mathcal{Z}_1 \rightrightarrows \mathcal{M}_1$ and $\mathcal{Z}_2 \rightrightarrows \mathcal{M}_2$ is a *groupoid morphism* if it is consistent with the projections and the compositions. More precisely, a morphism consists of two maps, $\Phi : \mathcal{Z}_1 \to \mathcal{Z}_2$ and $\phi : \mathcal{M}_1 \to \mathcal{M}_2$, such that

$$\alpha_2 \circ \Phi = \phi \circ \alpha_1 \qquad \beta_2 \circ \Phi = \phi \circ \beta_1, \tag{6}$$

and

$$\Phi(uv) = \Phi(u)\Phi(v) \qquad \forall u, v \in \mathcal{Z}_1. \tag{7}$$

In Equation (6), α_1, β_1 and α_2, β_2 denote the source and target maps of $\mathcal{Z}_1 \rightrightarrows \mathcal{M}_1$ and $\mathcal{Z}_2 \rightrightarrows \mathcal{M}_2$, respectively. It should be noted that, as a direct consequence of (7), a morphism automatically preserves the identities. Moreover, as a consequence of (6), the map ϕ is implied by the map Φ. A *groupoid isomorphism* is an invertible groupoid morphism whose inverse is also a groupoid morphism. A *Lie groupoid morphism* is a differentiable morphism between two Lie groupoids.

A *subgroupoid* $\mathcal{Z}' \rightrightarrows \mathcal{M}'$ of a groupoid $\mathcal{Z} \rightrightarrows \mathcal{M}$ is a groupoid such that $\mathcal{Z}' \subset \mathcal{Z}$ and $\mathcal{M}' \subset \mathcal{M}$ and such that the inclusion map is a groupoid morphism. A *Lie subgroupoid* is defined similarly, except that $\mathcal{Z}'$ and $\mathcal{M}'$ are required to be submanifolds of $\mathcal{Z}$ and $\mathcal{M}$, respectively, and the inclusion map is required to be a Lie groupoid morphism.

A transitive Lie subgroupoid $\mathcal{Z}'$ of a transitive Lie groupoid $\mathcal{Z}$ is said to be a *reduction*, if the base manifolds coincide. This terminology arises from considering that the only difference between the two groupoids is that the fibres have been curtailed. In other words, the structure group of the reduced subgroupoid is precisely a subgroup of the structure group of the original groupoid.

Remark 1 Given a groupoid $\mathcal{Z} \rightrightarrows \mathcal{M}$, and given a subset $\mathcal{U} \subset \mathcal{M}$, the set $\mathcal{Z}_{\mathcal{U}}^{\mathcal{U}} = \{\mathcal{Z}_X^Y \mid X, Y \in \mathcal{U}\}$ is a subgroupoid of $\mathcal{Z}$ with base $\mathcal{U}$. It is called the *restriction* of $\mathcal{Z}$ to $\mathcal{U}$, since it consists of only all the arrows that start and end in $\mathcal{U}$, the remaining arrows being discarded. Notice that if $\mathcal{Z}$ is transitive, so is $\mathcal{Z}_{\mathcal{U}}^{\mathcal{U}}$. In particular, for the material groupoid $\mathcal{P} \rightrightarrows \mathcal{B}$, the restriction $\mathcal{P}_{\mathcal{U}}^{\mathcal{U}}$ of $\mathcal{P}$ to a sub-body $\mathcal{U}$ is precisely the material groupoid of $\mathcal{U}$.

3.5 **The frame groupoid and homogeneity**

An important example of a transitive Lie groupoid defined on a manifold $\mathcal{M}$ is the *frame groupoid* $\mathcal{F}(\mathcal{M})$. Starting from the tangent bundle $T\mathcal{M}$, we consider each linear isomorphism $z : T_X\mathcal{M} \to T_Y\mathcal{M}$ between the tangent spaces at $X, Y \in \mathcal{M}$ as an arrow in the groupoid. The groupoid operation is the ordinary composition of maps. The importance of the frame groupoid in the context of Continuum Mechanics is that

Every material groupoid is a subgroupoid of the frame groupoid of $\mathcal{B}$. Moreover, if the body is smoothly uniform and its material groupoid is a Lie groupoid, the material groupoid is a reduction of the frame groupoid of the body.

The frame groupoid $\mathcal{F}(\mathcal{U})$ of an open subset $\mathcal{U} \subset \mathbb{R}^3$ plays an important role in the definition of the concept of *material homogeneity.* We note that, by the existence of a trivial translation, there is a natural Lie groupoid isomorphism between $\mathcal{F}(\mathbb{R}^3)$ and the Cartesian product $\mathbb{R}^3 \times \mathbb{R}^3 \times GL(3;\mathbb{R})$. Indeed, any arrow of $\mathcal{F}(\mathbb{R}^3)$ consists of a pair of points of $\mathbb{R}^3$ and a non-singular 3×3 matrix. For any (Lie) subgroup G of the general linear group $GL(3;\mathbb{R})$, we can accordingly generate a reduction of $\mathcal{F}(\mathcal{U})$ by means of its natural isomorphism with $\mathcal{U} \times \mathcal{U} \times G$. This reduction, which is clearly a transitive Lie groupoid, is called the *standard flat reduction* of $\mathcal{F}(\mathcal{U})$ to G.

To introduce the notion of homogeneity, we start by noting that the material groupoid $\mathcal{P}$ defined on the body manifold $\mathcal{B}$ is isomorphic (as a groupoid) to its representation in each configuration $\kappa : \mathcal{B} \to \mathbb{E}^3$. A uniform body is said to be *materially homogeneous* if, for some configuration κ, its material groupoid coincides with a standard flat reduction of $\mathcal{F}(\kappa(\mathcal{B}))$. In other words, the canonical translations in that configuration are material isomorphisms! A uniform body is *locally homogeneous* if it can be covered with homogeneous subbodies.

Remark 2 It is important to note that, since the material isomorphisms must of necessity transform according to the rules of change of configuration, the groupoid isomorphisms brought about by any changes of configurations are *lifts* of the respective deformations ϕ of the body, as shown schematically in Figure 3. By the term 'lift' we mean that the map Φ between the groupoid arrows is obtained from the derivative of the map ϕ between the set of objects. This is possible only because we are restricting attention to subgroupoids of the frame groupoid. Let $\mathbf{P} : T_X(\kappa_0(\mathcal{B})) \to T_Y(\kappa_0(\mathcal{B}))$ be a material isomorphism between points X and Y in the configuration κ_0. Let $\phi = \kappa_1 \circ \kappa_0^{-1}$ be the deformation between κ_0 and another configuration κ_1. The material isomorphism $\mathbf{P}'$ between the image points $X' = \phi(X)$ and $Y' = \phi(Y)$ corresponding to $\mathbf{P}$ is given by

$$\mathbf{P}' = \phi_*(Y) \circ \mathbf{P} \circ \phi_*^{-1}(X'). \tag{8}$$

To summarize,

Homogeneity of a smoothly uniform body $\mathcal{B}$ is tantamount to the existence of a configuration κ such that the induced material groupoid coincides with a standard flat reduction of the frame groupoid of $\kappa(\mathcal{B}) \subset \mathbb{E}^3$. Accordingly, homogeneity implies the existence of a configuration (unique modulo a rigid-body motion) whereby the trivial Euclidean translations are material isomorphisms. Intuitively, a uniform body is homogeneous if it can be 'straightened' so that its constitutive equation is independent of the (Cartesian) coordinates of its points. Local homogeneity requires only that this property be attainable chunk-wise.

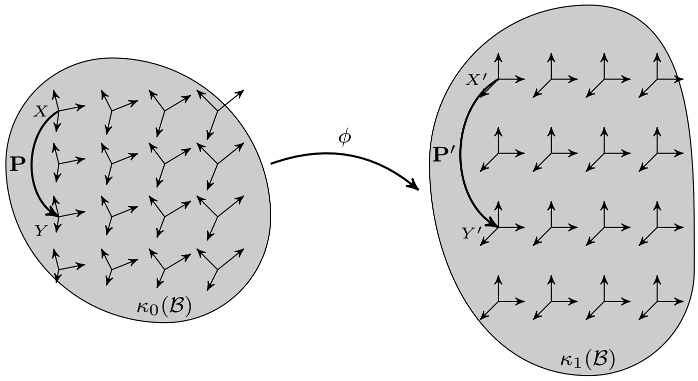

Fig. 3 Schematic depiction of homogeneity

4 Lie algebroids

4.1 Introduction

As an algebraic structure, a groupoid can be colourfully described as a group 'on steroids'. Indeed, a group can be regarded as the particular case of a groupoid whose set of objects $\mathcal{M}$ is a singleton. All the elements of a group can be mutually composed (multiplied) and there is a single, uniquely defined, unit element e. In a general groupoid $\mathcal{Z} \rightrightarrows \mathcal{M}$, on the other hand, $\mathcal{M}$ is an arbitrary set, but, except for the totally intransitive case, $\mathcal{Z}$ is not just the union of its individual vertex groups. The elements of $\mathcal{Z}$ are 'arrows' that may have different sources and targets. Arrows can be composed only if they satisfy the extra condition of being in tandem, tip-to-tail fashion. Moreover, rather than a single unit element, each element X of the set of objects $\mathcal{M}$ carries its own (unique) unit $i(X)$, an arrow in the form of a loop, so to speak.

Recall that a *Lie algebra* is a vector space endowed with an antisymmetric binary operation called a *Lie bracket*. Lie algebras are defined independently from groups, but the fundamental work of Sophus Lie (1842-1899) demonstrated the intimate connection that exists between Lie algebras and Lie groups, that is, groups that are also manifolds in which the operations of multiplication and inversion are smooth.

The Lie algebra of a Lie group represents an infinitesimal version of the latter in a precise sense. Its underlying vector space can be identified with the tangent space of the Lie group at the unit element. The vehicle to this identification is provided by the notion of left- (or right-) invariant vector fields on the Lie group. Similarly, the concept of a *Lie algebroid* can be introduced independently and eventually related to the notion of Lie groupoid. As an infinitesimal version of the latter, however, it involves certain tangent spaces to the groupoid $\mathcal{Z}$ at each of its unit elements. Again, these notions are intermediated via right- (or left-) invariant vector fields on the groupoid.

As everything else pertaining to groupoids, these notions acquire a further degree of sophistication as compared with their group counterparts. Although certainly premature for this introduction, we take the liberty of depicting, in Figure 4, a schematic drawing that may serve as an intuitive basis for a mental representation of the concepts that will be advanced below in a more precise fashion.

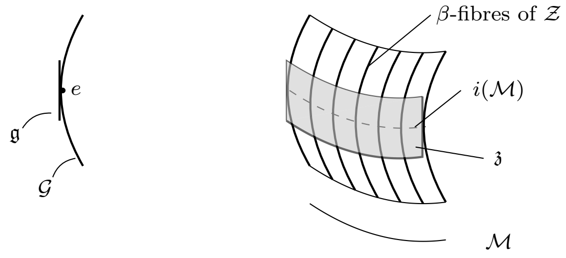

Fig. 4 A preliminary mental picture of the Lie algebroid $\mathfrak{z}$ of a Lie groupoid $\mathcal{Z}$ (right) as compared with the Lie algebra $\mathfrak{g}$ of a Lie group $\mathcal{G}$ (left). The β-fibre at $X \in \mathcal{M}$ is the collection of all the arrows arriving at X.

4.2 Definition

Let $\pi : A \to \mathcal{M}$ denote a *vector bundle* over a *base manifold* $\mathcal{M}$, and let $\Gamma(A)$ denote the space of its smooth global sections $\sigma : \mathcal{M} \to A$. A *Lie algebroid* structure on this vector bundle is obtained by specifying a bilinear *bracket* operation $[\cdot,\cdot] : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$ and a vector-bundle morphism $\sharp : A \to T\mathcal{M}$, called the *anchor map*. These maps must satisfy the following properties;

1. Skew-symmetry:
$$[\rho, \sigma] = -[\sigma, \rho] \qquad \forall \rho, \sigma \in \Gamma(A). \tag{9}$$
2. Jacobi identity:
$$[\rho, [\sigma, \tau]] + [\tau, [\rho, \sigma]] + [\sigma, [\tau, \rho]] = \mathbf{0} \qquad \forall \rho, \sigma, \tau \in \Gamma(A). \tag{10}$$
3. Consistency:
$$[\rho, \sigma]^\sharp = [\rho^\sharp, \sigma^\sharp] \qquad \forall \rho, \sigma \in \Gamma(A). \tag{11}$$
4. Leibniz rule:
$$[\rho, f\sigma] = f[\rho, \sigma] + \rho^\sharp(f)\, \sigma \qquad \forall \rho, \sigma \in \Gamma(A), f \in C^\infty(\mathcal{M}). \tag{12}$$

Remark 3 The first two properties are self-explanatory. The third property can be shown to be a consequence of the other ones. For compactness of notation, we have indicated by $\rho^\sharp$ the image $\sharp(\rho) \in \Gamma(T\mathcal{M})$ of the section $\rho \in \Gamma(A)$. Moreover, the bracket appearing on the right-hand side of (11) and (12) is the ordinary *Lie bracket* of vector fields in $T\mathcal{M}$. The fourth property requires some further clarification, as it displays the reason behind the need for an anchor map. In an arbitrary vector bundle, there is in principle no canonical action of the vectors in the bundle on a smooth real-valued function $f \in C^\infty(\mathcal{M})$ defined on the base manifold $\mathcal{M}$. It is only in the *tangent bundle* $T\mathcal{M}$ that such an action exists, providing us with the *directional derivative* $\mathbf{v}(f)$ of f in the direction of $\mathbf{v} \in T\mathcal{M}$.

A Lie algebroid is *transitive* if the anchor map $\sharp$ is a submersion.[2] It is *totally intransitive* if the anchor is the zero map (assigning to each vector in A the zero tangent vector at the corresponding point of the base manifold). The reason for this terminology will become apparent later.

4.3 The Lie algebroid of a Lie groupoid

4.3.1 The β-bundle

Consider the disjoint union $\mathcal{Z}^\mathcal{M}$ of all the β-fibres of a transitive Lie groupoid $\mathcal{Z}$, that is,

$$\mathcal{Z}^\mathcal{M} = \bigcup_{X \in \mathcal{M}} \mathcal{Z}^X. \tag{13}$$

This set, which we call the β-bundle, can be regarded as a fibre bundle over the base manifold $\mathcal{M}$ with projection β. In terms of arrows, $\mathcal{Z}^\mathcal{M}$ looks like a spider colony, each fibre $\mathcal{Z}^X$ being a spider with legs arriving at X and issuing from some point $Y \in \mathcal{M}$, as shown schematically in Figure 5 [5]. Notice that the total set of this fibre bundle is the same as the total set of the original transitive groupoid $\mathcal{Z}$. They both consist of the set of all arrows.

4.3.2 Left-invariant vector fields on a Lie groupoid

In any groupoid $\mathcal{Z} \rightrightarrows \mathcal{M}$ we can define the concept of *left translation*. Let $g \in \mathcal{Z}$ and let $z \in \mathcal{Z}$ be such that $\beta(z) = \alpha(g)$. The left translation of z by g is given by

$$L_g(z) = gz. \tag{14}$$

It follows from this definition that L_g maps β-fibres into β-fibres, that is, the spider at $\alpha(g)$ gets mapped to the fibre at $\beta(g)$. Specifically,

$$L_g : \mathcal{Z}^{\beta(z)} = \mathcal{Z}^{\alpha(g)} \to \mathcal{Z}^{\beta(g)}. \tag{15}$$

[2] Recall that a smooth map is a submersion if its derivative is surjective at each point.

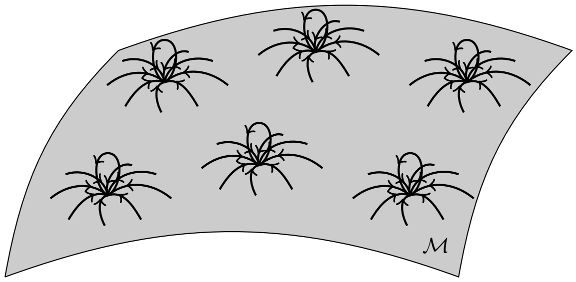

Fig. 5 The β-bundle $\mathcal{Z}^{\mathcal{M}}$ as a spider colony

If the groupoid happens to be a Lie groupoid, the left action is differentiable and we can consider the tangent map $(L_g)_* : T\mathcal{Z}^{\alpha(g)} \to T\mathcal{Z}^{\beta(g)}$. A vector field $V : \mathcal{Z} \to T\mathcal{Z}$ on $\mathcal{Z}$ is *left-invariant* if

$$(L_g)_*(V(z)) = V(L_g(z)) \qquad \forall g, z \in \mathcal{Z}. \tag{16}$$

Of necessity, a left-invariant vector field must be β*-vertical*, that is, it must dwell on the tangent spaces of the β-fibres of $\mathcal{Z} \rightrightarrows \mathcal{M}$.

A left-invariant vector field on a Lie groupoid is completely determined by its values at the unit elements $i(X)$, for all $X \in \mathcal{M}$. Indeed, setting $z = i \circ \alpha(g)$ in Equation (16), we obtain

$$(L_g)_*(V(i \circ \alpha(g))) = V(L_g(i \circ \alpha(g))) = V(g(i \circ \alpha(g))) = V(g). \tag{17}$$

Recall that, given any smooth vector field V on a manifold $\mathcal{M}$, the fundamental theorem of the theory of ODEs guarantees the existence and uniqueness of maximal smooth *integral curves* defined at each point of $\mathcal{M}$. If $\gamma_x = \gamma_x(t)$ is the integral curve containing the point $x \in \mathcal{M}$, the curve parameter can be adjusted by a mere translation such that $\gamma_x(0) = x$. By definition of integral curve, we have

$$V(x) = \left.\frac{d\gamma_x(t)}{dt}\right|_{t=0}. \tag{18}$$

Moreover, every smooth vector field acts as the *infinitesimal generator* of a *local flow* ϕ_t^V. For each t in a certain interval of $\mathbb{R}$ containing the origin, ϕ_t^V is a diffeomorphism of $\mathcal{M}$ defined by the prescription

$$\phi_t^V(x) = \gamma_x(t). \tag{19}$$

Clearly, $\phi_0^V = id_{\mathcal{M}}$ and $\phi_{-t}^V = \left(\phi_t^V\right)^{-1}$. Applying these concepts to Equation (16) we obtain the result of the following lemma.

Lemma 1 *A* β*-vertical vector field* V *on a Lie groupoid is left invariant if, and only if, its local flow commutes with left translations, that is,*

$$L_g \circ \phi_t^V(z) = \phi_t^V \circ L_g(z), \tag{20}$$

for all g, z *such that* $\alpha(g) = \beta(z)$.

4.3.3 The associated Lie algebroid

After the foregoing properties of left-invariant vector fields on a Lie groupoid $\mathcal{Z} \rightrightarrows \mathcal{M}$ have been established, we introduce the vector bundle $\pi : A\mathcal{Z} \to \mathcal{M}$ whose fibre at each $X \in \mathcal{M}$ is the tangent space to the β-fibre of $\mathcal{Z}$ at the identity $i(X)$. It is clear that a section of this vector bundle corresponds exactly to a left-invariant vector field on $\mathcal{Z}$. Since each map L_g is a diffeomorphism between two β-fibres, and since the left-invariant vector fields are tangent to these fibres, it follows that the ordinary Lie bracket between two left-invariant vector fields is again left-invariant. Therefore, given two sections of $A\mathcal{Z}$, we can define a Lie-algebroid bracket operation by considering the Lie bracket of the corresponding vector fields in $\mathcal{Z}$ and then considering its value at the unit section. To complete the determination of the Lie algebroid associated with the Lie groupoid $\mathcal{Z}$, we declare the anchor map to be given by $\alpha_* : T\mathcal{Z} \to T\mathcal{M}$. An intuitive idea of the anchor map (and of the meaning of the Lie algebroid) can be gathered from Figure 6. Starting from the identity loop-like arrow at a point $X \in \mathcal{M}$, we explore its vicinity in $\mathcal{Z}$ by keeping the tip of the arrow fixed at X, so as to stay always in the same β-fibre $\beta^{-1}(X)$. If we keep the tail of the arrow also at X (that is, if we explore just the loop-like neighbours), we are clearly moving within the vertex group at X. As a result, we recover the Lie algebra of this vertex group. In the case of the material groupoid $\mathcal{P}$ introduced above, we obtain the Lie algebra of the *material symmetry group* at X. Let us further explore the vicinity of the unit $i(X)$ by considering an arrow z with its tip at X, but with its tail elsewhere, at say $X + dX$. The differential projection $\alpha(z) - \alpha(i(X))$ is precisely dX. Thus, intuitively enough, we see how the map α_* acts as the anchor of the algebroid. We see, moreover, that the Lie algebra of the vertex group at X is precisely the kernel of the anchor at X. Moreover, if α_* is a surjective map, there are arrows between X and every point in an open neighbourhood of X in $\mathcal{M}$. This picture perfectly justifies the terminology introduced above for transitive and totally intransitive algebroids. In the case of the material groupoid $\mathcal{P}$ we conclude that a smoothly uniform body has a transitive *material Lie algebroid*.

5 The material Lie algebroid and homogeneity

5.1 Lie-algebroid morphisms and Lie subalgebroids

A morphism $\phi : A \to A'$ of two Lie algebroids $\pi : A \to \mathcal{M}$ and $\pi' = A' \to \mathcal{M}$ over a common base manifold $\mathcal{M}$ is defined as a vector-bundle morphism that is consistent with the respective anchors and brackets. That is, in an obvious notation,

$$\sharp = \sharp' \circ \phi, \tag{21}$$

and

$$\phi[\rho, \sigma] = [\phi \circ \rho, \phi \circ \sigma] \qquad \forall \rho, \sigma \in \Gamma(A). \tag{22}$$

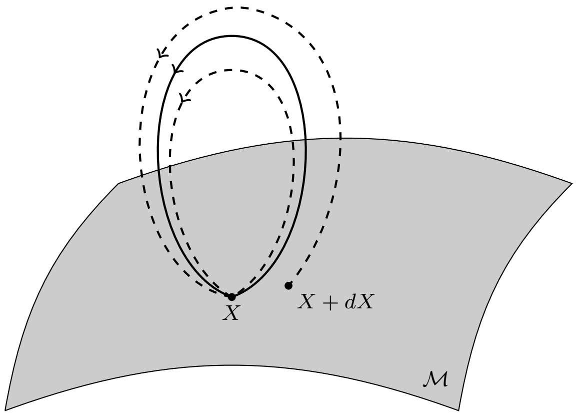

Fig. 6 Intuiting the Lie algebroid of a Lie groupoid. The identity $i(X)$ is drawn as a solid arrow, while elements in its vicinity are drawn as dashed arrows. The anchor map assigns to each dashed arrow the opening between its tail and X.

Let A' be a vector subbundle of A, and let the inclusion map $incl : A' \hookrightarrow A$ be a Lie-algebroid morphism. We say that A' is a *Lie subalgebroid* of A. When both A and A' are transitive Lie algebroids, A' is called a *reduction* of A.

Consider the frame groupoid $\mathcal{F}(\mathbb{R}^3) \approx \mathbb{R}^3 \times \mathbb{R}^3 \times GL(3;\mathbb{R})$, which we have already encountered. The β-fibre at $X \in \mathbb{R}^3$ consists of elements of the form $(\{Y\}, \{X\}, [A])$, where $\{Y\}$ is a triple containing the coordinates of the variable tail and $\{X\}$ are the coordinates of the common tip X, and where $[A]$ is a 3×3 non-singular matrix. A vector tangent to this fibre will, therefore, have the component form $(\{v\}, \{X\}, [B])$, where $\{v\}$ are the 3 components of a tangent vector to $\mathbb{R}^3$ at X and where $[B]$ is an arbitrary 3×3 matrix. The same reasoning applies to the standard flat reduction of $\mathcal{F}(\mathbb{R}^3)$ to a subgroup $G \subset GL(3;\mathbb{R})$. In other words, the Lie algebroid associated to this reduction is $T\mathbb{R}^3 \oplus \left(\mathbb{R}^3 \times \mathfrak{g}\right)$, where $\mathfrak{g}$ denotes the Lie algebra of G. In the imagery suggested by Figure 6, the first component corresponds to the opening of arrows with tip at X, while the last component is spanned by al the loop-like arrows at X, namely, the Lie algebra of the symmetry group.

5.2 Homogeneity and material connections

Following the lead of the groupoid treatment counterpart, we assert that a smoothly uniform body $\mathcal{B}$ is homogeneous if its (transitive) material Lie algebroid $A\mathcal{P}$ in some configuration κ coincides with the Lie algebroid of a standard flat reduction of the frame groupoid of (an open subset of) $\mathbb{R}^3$. Local homogeneity is defined accordingly.

Our intention in this section is to demonstrate the relation between this definition of homogeneity and the notion of *material connections* introduced by Noll [14] and Wang [16]. As a first step in that direction, consider a transitive

Lie algebroid $\mathcal{Z} \rightrightarrows \mathcal{M}$ and arbitrarily fix a *reference or archetypal point* $X_0 \in \mathcal{M}$. It is not difficult to verify that the disjoint union

$$\mathcal{Z}_0 = \bigcup_{X \in \mathcal{M}} \mathcal{Z}_{X_0}^X \tag{23}$$

is a principal fibre bundle with structure group $\mathcal{G}$ equal to the vertex group at X_0 and with bundle projection β. A connection in a principal bundle is a *horizontal distribution*, namely, the point-wise specification of a non-vertical subspace $\mathcal{H}$ such that (i) the tangent space is the direct sum of $\mathcal{H}$ and the (vertical) tangent space to the fibre; (ii) the horizontal distribution is invariant under the right action of the structure group.

Let $\Lambda : T\mathcal{M} \to A\mathcal{Z}$ be a vector-bundle morphism such that $\sharp \circ \Lambda = id_{T\mathcal{M}}$. One can directly verify that there is a one-to-one correspondence between such vector-bundle morphisms and connections in the principal bundle $\mathcal{Z}_0$. In the context of Noll's formulation, local homogeneity is equivalent to the existence of a (locally defined, say) material connection with vanishing torsion. We conclude, therefore, that local homogeneity is guaranteed by the existence of a (local) vector-bundle morphism from the (restricted) material Lie algebroid to the tangent bundle of the (sub-)body such that the associated connection is torsionless.

6 The material distribution

6.1 General considerations

The material groupoid is essentially an algebraic entity based on the establishment of an equivalence relation between body points made of the same material. This algebraic entity has also a definite topological and geometric flavour arising from the fact that the elements of a groupoid can be visualized as arrows connecting points. A differential geometric aspect is added when the material groupoid is a Lie groupoid. In this last case, the study of its associated *Lie algebroid* is also of great relevance, particularly in the study of local homogeneity conditions.

As already mentioned in Section 3.3, even when the constitutive response is a perfectly smooth function, the resulting material groupoid is not necessarily a Lie groupoid. This somewhat surprising fact leads us to introduce and investigate a new differential geometric construct, called the *material distribution*, that answers the question as to what is the most general differentiable structure that can most faithfully reflect the consequences of having a smooth constitutive response even in the absence of material uniformity.

A heuristic clue is gathered by thinking of the constitutive law ψ as a function defined over the product $\kappa_0(\mathcal{B}) \times GL(3, \mathbb{R})$ of the body manifold in a reference configuration κ_0 times the general linear group (of non-singular matrices) in 3 dimensions. Appropriately, the function ψ assigns to each material point and each deformation gradient a value of the constitutive quantity.

Consider now a parametric curve $\gamma(t)$ in $\mathcal{B}$ such that all the points along this curve are smoothly materially isomorphic, so that there exists a (not necessarily unique) smooth field $\mathbf{P}(t)$ of material isomorphisms from the point $\gamma(0)$ to the points $\gamma(t)$, such that $\mathbf{P}(0) = \mathbf{I}$. This means that the function ψ, when restricted to the curve, can be expressed as $\psi(\gamma(t), \mathbf{F}) = \psi(\gamma(0), \mathbf{FP}(t))$. For each fixed value of the deformation gradient $\mathbf{F}$ we can *lift* the curve $\gamma(t)$ to a curve $\hat{\gamma}(t)$ in $\kappa_0(\mathcal{B}) \times GL(3, \mathbb{R})$ by using the value of $\mathbf{FP}(t)$ as the coordinate along $GL(3, \mathbb{R})$. By the assumed material isomorphism, the value of ψ will remain constant along $\hat{\gamma}(t)$. Moreover, for two different values of $\mathbf{F}$, the curves will be "parallel" in the sense that they differ only by multiplication of their matrix argument to the left by a different fixed factor $\mathbf{F}$, while $\mathbf{P}(t)$ remains unchanged.

Since a small piece of a curve is, in the limit, a tangent vector, the description just outlined leads us to consider the collection of all vector fields in $\kappa_0(\mathcal{B}) \times GL(3, \mathbb{R})$ that are left invariant under the action of $GL(3, \mathbb{R})$ and, at the same time, annihilate the differential of the constitutive function ψ. This approach was adopted in [9]. Here, we generalize this construction by working directly on the frame groupoid $\mathcal{F}(\mathcal{B})$, rather than on the product $\kappa_0(\mathcal{B}) \times GL(3, \mathbb{R})$. The local vector fields enjoying the properties outlined above can be regarded as the generators of a unique *singular distribution* in $\mathcal{F}(\mathcal{B})$ completely characterizing the smoothness of the constitutive law of departure and its physical consequences in terms of a unique partition of the body into disjoint smoothly uniform components.

6.2 **Coordinate expressions**

For the sake of clarity, let us record the representation of a left-invariant vector field on the frame groupoid of a manifold $\mathcal{M}$ in terms of local coordinates induced by a reference configuration κ_0. A variable element a in the β-fibre at a point $Y \in \mathcal{M}$ has coordinates $\left(Y^I, \left(X^L, A^J_K\right)\right)$, where X^L are the coordinates of $X = \alpha(a) \in \mathcal{M}$. The unit element of this fibre has coordinates $\left(Y^I, \left(Y^L, \delta^J_K\right)\right)$. Let $b \in \mathcal{F}(\mathcal{M})$ be an element such that $\alpha(b) = \beta(a)$, and let its coordinates, as an element of the β-fibre at $Z = \beta(b)$, be $\left(Z^I, \left(Y^L, B^J_K\right)\right)$. The left action by b, as an element of the β-fibre at $\beta(b)$ has the coordinate representation

$$L_b(a) = \left(Z^I, \left(X^L, B^J_M A^M_K\right)\right). \tag{24}$$

A tangent vector V to the β-fibre at Y is expressed in components as

$$V\left(Y^I, \left(X^L, A^J_K\right)\right) = \left(Y^I, \left(X^L, A^J_K\right), 0, v^M, v^N_Q\right). \tag{25}$$

This vector is mapped to the vector

$$(L_b)_*\left(V\left(Y^I, \left(X^L, A^J_K\right)\right)\right) = \left(Z^I, \left(X^L, B^J_S A^S_K\right), 0, v^M, B^N_S v^S_Q\right). \tag{26}$$

For a left-invariant vector field this vector should be the same as the value of the vector field at the point $\left(Z^I, \left(X^L, B^J_S A^S_K\right)\right)$. Let the vector field attain values at the units given in coordinates by

$$V\left(Y^I,\left(Y^L,\delta^J_K\right)\right)=\left(Y^I,\left(Y^L,\delta^J_K\right),0,w^M(Y),w^N_Q(Y)\right). \tag{27}$$

Left-invariance demands that

$$V\left(Z^I,\left(Y^L,B^J_K\right)\right)=\left(Z^I,\left(Y^L,B^J_K\right),0,w^M(Y),B^N_S w^S_Q(Y)\right). \tag{28}$$

6.3 **Constitutive invariance**

In a reference configuration κ_0, a constitutive response such as (3) is expressed as a function

$$\psi=\psi(F^i_I,X^J). \tag{29}$$

It is convenient, however, to reinterpret this as a function $\tilde{\psi}$ defined over the frame groupoid $\mathcal{F}(\mathcal{M})$. This interpretation is possible because, having chosen a reference configuration κ_0, the maps between tangent spaces can be regarded as deformation gradients $\mathbf{F}$ if so desired. At the element $\left(Y^I,\left(X^L,A^J_K\right)\right)$ of the β-fibre at Y, the constitutive response is given by

$$\tilde{\psi}\left(Y^I,\left(X^L,A^J_K\right)\right)=\psi(A^J_K,X^L). \tag{30}$$

We have exploited the fact that the constitutive response at a point does not depend on the target point, since it is invariant under translations in $\mathbb{E}^3$. The differential of the function $\tilde{\psi}$ is

$$d\tilde{\psi}=\frac{\partial\psi}{\partial X^L}dX^L+\frac{\partial\psi}{\partial A^J_K}dA^J_K. \tag{31}$$

We are interested in finding the collection $\mathcal{V}$ of all the smooth local vector fields V that satisfy three conditions: (a) they are everywhere tangent to the β-fibres; (b) they annihilate the differential of the constitutive response; and (c) they are left-invariant. Condition (a) is implied by (c). We shall call these vector fields *constitutively admissible*, or just *admissible*. It is certainly possible to proceed analytically so as to identify exactly what these vector fields represent from the point of view of continuum mechanics. Instead, at the risk of some loss of rigour, we will advance a more geometric argument.

We commence by noticing that condition (b) implies that these vector fields are, by construction, everywhere tangent to the level sets of the function $\tilde{\psi}$. As a consequence of this simple observation, we conclude that the integral curves of the admissible vector fields are entirely contained in these level sets. By condition (a), the desired integral curves belong to the intersection of the level sets of $\tilde{\psi}$ with the β-fibres. Moreover, because of the already noticed independence of $\tilde{\psi}$ on the target point, we observe that the picture of the integral-curve-portraits in each and every β-fibre are all identical. Finally, the left invariance condition (c) means that, under the action of an arbitrary element z of the frame groupoid, integral curves are mapped into integral curves.

If $z(t)$ is the parametric representation of an integral curve of $\mathcal{V}$, by condition (a) we have necessarily $\beta(z(t)) =$ constant. On the other hand, the projected curve $\alpha(z(t))$ consists entirely of mutually materially isomorphic points in $\mathcal{B}$. Indeed, by conditions (b) and (c) above, they respond in "parallel" ways to all possible left translations in the frame groupoid. The main conclusion of this observation is that the set of integral curves of $\mathcal{V}$ and the material groupoid are intimately related according to the following statement.

If the material groupoid has a point (that is, an arrow z) in common with an integral curve of $\mathcal{V}$, then the whole integral curve belongs to the material groupoid.

6.4 Singular distributions

A *distribution* $\mathcal{D}$ over a manifold M is obtained by assigning to each point $x \in M$ a vector subspace D_x of the tangent space T_xM. The distribution is defined as the disjoint union

$$\mathcal{D} = \bigcup_{x\in M} D_x. \tag{32}$$

The distribution $\mathcal{D}$ is said to be *regular* or of *constant rank* if the dimension of D_x is the same for all $x \in M$. Otherwise, the distribution is called *singular* [13]. Consider the collection $\mathcal{V}_{\mathcal{D}}$ of all local vector fields that belong to the distribution. The distribution $\mathcal{D}$ is *smooth* if it is spanned by $\mathcal{V}_{\mathcal{D}}$. Equivalently, for every $x \in M$, D_x coincides with the set of all linear combinations of all vectors $V(x)$ of all local vector fields $V \in \mathcal{V}_{\mathcal{D}}$ defined at x.

An *integral manifold* of a distribution $\mathcal{D}$ in M is an immersed submanifold $N \subset M$ such that, at each point $x \in N$, $T_xN = D_x$. In the case of a regular one-dimensional distribution, every point is contained in an integral manifold. A distribution of higher dimension need not have any integral manifolds. A *maximal integral manifold* is an integral manifold that is not contained in any strictly large integral manifold. Given a distribution on M, it can be shown that each point $x \in M$ that is contained in some integral manifold is contained in a unique maximal integral manifold.

A smooth distribution $\mathcal{D}$ on a manifold M is said to be *integrable* if every point $x \in M$ is contained in an integral manifold of $\mathcal{D}$ and, hence, in a unique maximal integral manifold. An integrable distribution induces a partition of the manifold M into its maximal integral manifolds. This partition is known as a *foliation* of M, with each maximal submanifold referred to as a *leaf* of the foliation. Notice that in a strictly singular integrable distribution there will be leaves with different dimensions. To emphasize this fact, we sometimes refer to the partition induced by a singular integrable distribution as a *singular* foliation. The following theorem establishes necessary and sufficient conditions for integrability of a singular distribution.

Theorem 1 *(Stefan-Sussman) Let $\mathcal{D}$ be a smooth singular distribution on a smooth manifold M. Then the following three conditions are equivalent:*

1. *$\mathcal{D}$ is integrable;*
2. *$\mathcal{D}$ is spanned by a family of local vector fields, with respect to which it is invariant;*
3. *$\mathcal{D}$ is the tangent distribution of a smooth singular foliation.*

Condition 2 generalizes to singular distributions the condition of involutivity in the classical Frobenius theorem, which is valid for regular distributions. For singular distributions, involutivity is insufficient to guarantee integrability. What may fail is not involutivity but *stability*, which is the meaning of condition 2 above. A distribution is stable if it is invariant under the flow of any of its generating vector fields.

6.5 **The material distribution and its body projection**

The collection $\mathcal{V}$ of admissible vector fields introduced in Section 6.3 generates, by considering all point-wise linear combinations, a smooth singular distribution, denoted by $A\mathcal{P}$, called the *material distribution*. The following proposition is crucial to obtain relevant physical results.

Proposition 1 *The material distribution $A\mathcal{P}$ is integrable.*

Proof According to Theorem 1, we only need to prove that the image of any admissible vector field by the flow of any other admissible vector field is again an admissible vector field. Let V and W be two admissible vector fields and denote by W' the image of W by the (derivative of) the flow of V. By definition of derivative, the field W' is everywhere tangent to the image of the integral curves of W by the flow of V. At a point $z \in \mathcal{F}(\mathcal{B})$, the mapped integral curve is, accordingly, given by

$$\phi_s^{W'}(z) = \phi_t^V \circ \phi_s^W \circ \phi_{-t}^V(z). \tag{33}$$

Since the vector fields of departure, V and W, are left invariant, their respective flows commute with the left translation and, therefore, so does their composition on the right-hand side of Equation (33). Consequently, the flow of W' also commutes with left translations. By Lemma 1, we conclude that W' is left-invariant. Moreover, since the function $\tilde{\psi}$ is constant on each integral curve, it is also constant on the integral curves of W', so that the field W' is admissible.

Having just proved that the material distribution is always integrable, we can invoke the third part of Theorem 1 to conclude that there exists a smooth singular foliation $\mathcal{S}$ in $\mathcal{F}(\mathcal{B})$ such that $A\mathcal{P}$ is its tangent distribution. This fact has important ramifications, stemming from the fact that it can be shown that each leaf of the foliation is entirely contained in a β-fibre of $\mathcal{F}(\mathcal{B})$. Moreover, if $z \in \mathcal{P}$ then the leaf containing z is entirely contained in a β-fibre of $\mathcal{P}$. We call the collection of all the leaves of $\mathcal{S}$ that are contained in $\mathcal{P}$ the *material foliation*, denoted by $\mathcal{S}(\mathcal{P})$. This terminology is slightly misleading, since $\mathcal{S}(\mathcal{P})$ is technically not a foliation of $\mathcal{P}$, which may not even be a manifold.

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