REDUCTION OF VARIABLES IN THE SEARCH FOR CONDITIONAL END OF MULTIVARIANT FUNCTIONS. HYDRAULIC APPLICATION

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SUMMARY / ABSTRACT

The traditional method of multipliers or Lagrange's operators to the resolution of the problems in conditional ends of several variables, or the jacobians determinants, they are only needed in the presence of points of chair (or "tack") either when the implied form of the constraint prevents clear or variables that you want to replace in the objective function to optimize. It can also happen that expressed methods do not provide definitive solutions and have to resort, precisely, to the aforementioned technique of reduction or elimination of variables to effectively solve the problem, as we will have opportunity to see. For greater clarity in the process, develop several representative exercises and a practical case of the advantages offered by the technology in question.

Key words: ends, determinant equation, objective function, Lagrange's operator, independent variable, functional determinant, critical point.

RESUMEN

El método tradicional de los multiplicadores u operadores de Lagrange para la resolución de los problemas de extremos condicionados de varias variables, o el de los determinantes jacobianos, son sólo necesarios en presencia de puntos de silla (o de "ensilladura") o bien cuando la forma implícita de la restricción impide despejar la o las variables que interese substituir en la función objetivo a optimizar. Puede suceder, también, que los expresados métodos no ofrezcan soluciones definitivas y haya que recurrir, justamente, a la técnica referida de reducción o eliminación de variables para solventar eficazmente el problema planteado, como tendremos ocasión de comprobar. Para una mayor claridad del proceso, se desarrollan varios ejercicios representativos y un caso práctico de las ventajas que ofrece la técnica en cuestión.

***Palabras clave**: extremos, ecuación condicionante, función objetivo, operador de Lagrange, determinante funcional, variable independiente, punto crítico.*

INTRODUCTION

A problem that occurs frequently in Mathematical Analysis is to determine the relative or local extremes (maximum and/or minimum) of a real function whose real variables are not independent but are linked by one or more conditioning equations. We say, then, that it is a problem of "tied or conditioned extremes".

They usually appear in some problems of Physics, Economics or Engineering, as will be seen in the Hydraulics case study that appears at the end of this article. Thus, the traditional method of Lagrange multipliers or operators, or that of Jacobian determinants, are only necessary in the presence of saddle points (or "saddling") or when the implicit form of the restriction prevents clearing the variables that you want to replace in the objective function to be optimized. It may also happen that the aforementioned methods do not offer definitive solutions and it is necessary to resort, precisely, to the aforementioned technique to effectively solve the problem posed, as we will have occasion to verify.

In effect, suppose that the conditioning equation allows us to clear one of the variables as a function of the others, for example, in the form: $z = \Phi(x, y)$, and substituting it in the objective function to be optimized we obtain:

$$u = f\,[x,y,\ \Phi(x, y)\,] = F(x,y),$$

and the problem will be to look for the extreme values of F(x, y) whose variables are already independent, for which the established classical criteria can be applied.

Well, since it is a question that, generally, is not expressly contemplated in the mathematical analysis treaties in use, we have considered it convenient to develop it with the support of some general examples and a practical case of application of fluid mechanics that we judge sufficiently representative in this regard.

METHODOLOGY AND THEORETICAL BASIS

Lagrange Operators Method

Let the function z = f(x,y) be subject to the condition g(x,y) = 0. To obtain the maximum or minimum conditioned, the Lagrange function is formed:

$$\phi(x,y) = f(x,y) + \lambda\cdot g(x,y).\ \text{Thus:}$$

- *Necessary or first degree condition*:

The ends sought are from the system of equations:

$$\begin{cases} \dfrac{\partial\phi(x,y)}{\partial x} = f'_x(x,y) + \lambda \cdot g'_x(x,y) = 0 \\ \dfrac{\partial\phi(x,y)}{\partial y} = f'_y(x,y) + \lambda \cdot g'_y(x,y) = 0 \end{cases}$$

- *Sufficient or second degree condition*:

Now forming the second differential:

$$d^2\phi(x,y) = \frac{\partial^2\phi}{\partial x^2}dx^2 + 2\frac{\partial^2\phi}{\partial x\partial y}dxdy + \frac{\partial^2\phi}{\partial y^2}dy^2 \text{, with the condition:}$$

$$\frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy = 0 .$$

Thus, the function has a maximum if $d^2\phi < 0$ and a minimum if $d^2\phi > 0$ (García and Rodríguez, 1985). If $d^2\phi = 0$ is a doubtful case and further investigation is required.

This second degree or order condition can be discriminated, frequently, through the formation of the so-called "relevant Hessian border", which offers the following values:

$$H(x,y,\lambda) = \begin{vmatrix} \Phi''_{x^2} & \Phi''_{xy} & \Phi''_{x\lambda} \\ \Phi''_{xy} & \Phi''_{y^2} & \Phi''_{y\lambda} \\ \Phi''_{x\lambda} & \Phi''_{y\lambda} & \Phi''_{\lambda^2} \end{vmatrix} \left\langle \begin{array}{l} > 0 \rightarrow \text{MAXIMUM (2 variables)} \\ < 0 \rightarrow \text{MINIMUM} \end{array} \right.$$

where always $\Phi''_{\lambda^2} = 0$. This process generalizes *n* variables, like this:

MINIMUM → always H < 0.

MAXIMUM → 3 variables → H < 0 → u = f(x, y, z)
4 variables → H > 0 → u = f(x, y, z, t)
5 variables → H < 0 → u = f(x, y, z, t, s)
…………………(and so on and so on). With H = 0 it is a doubtful case and we must continue investigating.

In most practical problems it is not necessary to make this distinction, since at first sight the nature of the extreme or critical point in question is known.

Jacobian determinants method

Let, in the case of 2 variables, be the objective function: z = f(x,y) and the condition equation: g(x,y) = 0. The system:

$$\begin{cases} z = f(x,y) \\ g(x,y) = 0 \end{cases} \qquad [I]$$

In general, it will represent a curve in the Euclidean three-dimensional affine space and the values that z takes will be those of the function **f** along the curve **g**. Therefore, reasoning as it is done to obtain the ordinary extremes, the necessary condition for the existence of a conditioned extreme at a point will be the nullification, at that point, of z'_g. Then, to obtain the possible extreme points, we will form the system:

$$\begin{cases} z'_s = \dfrac{\dfrac{\partial(f,g)}{\partial(x,y)}}{\sqrt{g'^2_x + g'^2_y}} = 0 \\ g(x,y) = 0 \end{cases}$$

Or, since $\sqrt{g'^2_x + g'^2_y}$ it is always positive (note that both derivatives, if the system [I] represents a curve, are not identically null, simultaneously) the system is equivalent to:

$$\begin{cases} \dfrac{\partial(f,g)}{\partial(x,y)} = J(x,y) = 0 \\ g(x,y) = 0 \end{cases}$$

where J(x,y) is the Jacobian functional determinant.

To deduce the sufficient conditions, it will suffice to study the sign of z''_{g^2}. Remembering that:

$$z'_g = \frac{\dfrac{\partial z}{\partial x}\cdot g'_y - \dfrac{\partial z}{\partial y}\cdot g'_x}{\sqrt{g'^2_x + g'^2_y}} = \frac{J}{\sqrt{g'^2_x + g'^2_y}}$$, you will have to: $$z''_{g^2} = \frac{\dfrac{\partial z'_g}{\partial x}\cdot g'_y - \dfrac{\partial z'_g}{\partial y}\cdot g'_x}{\sqrt{g'^2_x + g'^2_y}},$$

that, at the points where J(x,y) is canceled, it becomes:

$$z''_{g^2} = \frac{\sqrt{g'^2_x + g'^2_y}\cdot J'_x\, g'_y - \sqrt{g'^2_x + g'^2_y}\cdot J'_x\, g'_y}{(g'^2_x + g'^2_y)^{3/2}} = \frac{\dfrac{\partial(J,g)}{\partial(x,y)}}{g'^2_x + g'^2_y}.$$

Therefore, if at one point of those found:

$$\begin{cases} z'' > 0 \rightarrow \dfrac{\partial(J,g)}{\partial(x,y)} > 0\text{, there is a relative minimum} \\ z'' < 0 \rightarrow \dfrac{\partial(J,g)}{\partial(x,y)} < 0\text{, there is a relative maximum} \end{cases}$$

NOTE: The exposed method is generalizable to *n* variables $x_1, x_2, ..., x_n$; if we try to obtain the extremes of $z = f(x_1, x_2, ..., x_n)$ with the (n-1) constraints:

$$g_1(x_1, x_2, \ldots, x_n) = 0 \; ; \; g_2(x_1, x_2, \ldots, x_n) = 0 \; ; \; \ldots ; \; g_{n-1}(x_1, x_2, \ldots, x_n) = 0 \; ;$$

the system formed by the following *n* equations is solved:

$$J = \frac{\partial(f, g_1, g_2, ..., g_{n-1})}{\partial(x_1, x_2, ..., x_n)} = 0 \; ; \; g_1 = 0 \; ; \; g_2 = 0 \; ; \; \ldots ; \; g_{n-1} = 0 \; ;$$

and to determine if it is a maximum or a minimum, the sign of is calculated in each one of the points found:

$$J_1 = \frac{\partial(J, g_1, g_2, ..., g_{n-1})}{\partial(x_1, x_2, ..., x_n)} \; ,$$

resulting in a local maximum if $J_1 < 0$ and a local minimum if $J_1 > 0$ (García and Rodríguez, 1985).

Variable substitution, elimination or reduction method

The problem of the conditioned extremes, generalized to *n* variables, is to find the extremes of the function $z = f(x_1, x_2, ..., x_n)$ that satisfy the conditioning equation: $g(x_1, x_2, ..., x_n) = 0$. If it is possible to solve this last equation for one of the variables, such as: $x_1 = h(x_2, \ldots, x_n)$, the solution of x_1 can be substituted in z resulting in: $f[h(x_2, \ldots, x_n), x_2, ..., x_n]$, which is a function of (n-1) variables. Let's call this function $F(x_2, ..., x_n)$; Obtaining the extremes of z, subject to the condition g previously expressed, is equivalent to obtaining the unconditioned extremes of $F(x_2, ..., x_n)$ with respect to the variables $x_2, ..., x_n$. The problem of conditioned extremes is thus reduced to that of an unconditioned one and with the same variables or one less variable, which we can solve in the usual way. That is, it allows us to go from a restricted optimization program with equality restrictions to a free classical optimization without any restrictions and with the same or less number of variables, which greatly simplifies the resolution process.

On the contrary, this procedure can be attributed to involving a loss of symmetry because it gives preference to one of the variables of the condition (which, normally, will be the easiest to clear based on the others). In any case, to be able to carry out the aforementioned substitution in a general problem of this type, it must be possible to explain *m* decision variables based on the remaining (n - m), which is the number of degrees of freedom of the problem posed. And this is not always possible, although it is possible in the vast majority of practical problems, which is why it is presented here through some explanatory examples that we will see below.

It is always assumed that the number of variables *n* and the number of constraints *m* are finite, and also n > m. If it happens that n < m, it may turn out that the set of feasible solutions is the vacuum or infinity, with which there is no solution, or the optimization problem is trivial.

On the other hand, the equality constraints in an optimization problem "reduce" its dimension. In general, for each restriction that is added, a degree of freedom is lost when obtaining the values that make the objective function reach its optimal value (Guzmán *et alt.*, 1999).

Interpretation of the Lagrange multipliers

In previous sections it has been seen that the points obtained when solving a program with equality constraints have associated so-called Lagrange multipliers (one for each constraint or conditioning equation). We will refer, next, to the meaning of these multipliers, of special importance in their different applications (Balbás and Gil, 2004).

To do this, let's formulate a program with equality restrictions, like the following:

$$\text{Optimize: } f(x_1,\ldots,x_n)$$

$$\text{subject to: } \left.\begin{array}{l} g_1(x_1,\ldots,x_n) = b_1 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ g_m(x_1,\ldots,x_n) = b_m \end{array}\right\} \quad [II]$$

where f: A $\rightarrow$ R and g = (g_1,…, g_m): A$\rightarrow$ R^m (n > m) are two functions of class C^2 in the open $A \subset R^n$.

If we assume that b_1, ..., b_m, can vary, it is clear that the feasible set M will depend on b = (b_1, ..., b_m), and we will write symbolically:

$$M(b) = \{x \in A \,/\, g(x) = b\}.$$

Intuitively, it is clear that the optimal points of the program [II] will depend on the value of b = (b_1, ..., b_m). Thus if given $\bar{b} = (\bar{b}_1,\ldots,\bar{b}_m)$ the program [II] has an optimum at the point: $(\bar{a},\bar{\lambda})(\bar{a} = (\bar{a},\ldots,\bar{a}_n),\bar{\lambda} = (\bar{\lambda}_1,\ldots,\bar{\lambda}_m))$, and we can establish a function F: B $\rightarrow$ R (B is an environment of b), such that F (b) = f (a) $\forall b \in B$, is already the optimum of the program for $b \in B$.

Well, given a program like [II], if for $\bar{b} = (\bar{b}_1,\ldots,\bar{b}_m)$ the function *f* it has a relative extreme on the set M(b) at the point $(\bar{a},\bar{\lambda})$ where the Jacobian functional determinant of the function **g** has range **m** and the Ordered Hessian determinant is not null, then it is true that:

$$-\lambda_i = \frac{\partial F(b)}{\partial b_i}.$$

Thus, this multiplier associated with the i-th restriction, measures the rate of variation of the value of the objective function *f* at the optimal point with respect to its corresponding b_i. The opposite of the k-th Lagrange multiplier measures the marginal change in the optimal value of the objective function with respect to the variation of the independent term of the k-th restriction b_k. That is to say:

$$\frac{\partial f(z^*)}{\partial b_k} = -\lambda_k^* .$$

But the Lagrange multipliers λ_i ($\forall$ i = 1, 2, ..., m) can make some sense. We have shown that Lagrange multipliers are equivalent to partial derivatives; and these derivatives are synonymous with the term "marginal". Therefore, λ_i multipliers can be interpreted as marginal changes (Sánchez, 2014).

SOME ILLUSTRATIVE EXAMPLES

Example 1

Find, by the Lagrange multipliers method, the conditioned extremes of the function: $z = x \cdot y$, if $x + y = 1$.

a) The Lagrange or *Lagrangian* function is:

$\phi(x,y) = x \cdot y + \lambda(x + y - 1)$, from where:

- Necessary or first degree condition:

$$\frac{\partial \phi}{\partial x} = y + \lambda = 0\,;\ \frac{\partial \phi}{\partial y} = x + \lambda = 0\,;\ x + y = 1,$$

from where, for $\lambda = -\frac{1}{2}$, we obtain: $x = y = \frac{1}{2}$, $z = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.

- Sufficient or second degree condition:

To determine if it is maximum or minimum, we will do:

$$\frac{\partial^2 \phi}{\partial x^2} = 0\,;\ \frac{\partial^2 \phi}{\partial x \partial y} = (\text{motto of Schwartz}) = \frac{\partial^2 \phi}{\partial y \partial x} = 1\,;\ \frac{\partial^2 \phi}{\partial y^2} = 0\,,$$

then, substituting: $d^2\phi = 0 \cdot dx^2 + 2 \cdot 1 \cdot dx \cdot dy + 0 \cdot dy^2$,

and as: $x + y = 1$, $dx + dy = 0$, that is, $dy = -dx$.

Substituting again, you will have to:

$$d^2\phi = 2\cdot dx\cdot dy = 2\cdot dx\,(-dx) = -2\cdot dx^2 < 0.$$

Since $d^2\phi$ is negative, at point P(½,½,¼) there is therefore a relative or local maximum.

b) Another rather more immediate way of solving it, by the variable reduction method advocated here, would lead to the following single-variable function:

Like: $y = 1 - x$; $z = x\,(1 - x) = x - x^2$; and so:

- Necessary or first degree condition:

$$z'_x = 1 - 2x = 0;\ 2x = 1;\ x = ½\ ;\ \ y = 1 - ½ = ½\ .$$

Similarly: $z = ½ \cdot ½ = ¼$.

- Sufficient or second degree condition:

And as: $z''_{x^2} = -2 < 0$ $\rightarrow$ local MAXIMUM local, reaching the same conclusion, although in a simpler way, than operating by the previous procedure.

Example 2

Find the relative extremes of: $z = x\cdot y^2$, if $x + y = 6$, using the Lagrange multiplier method.

a) The Lagrange or auxiliary function is, in this case:

$$\phi(x,y) = x\cdot y^2 + \lambda(x + y - 6).$$

- *Necessary or first degree condition*:

Partially deriving: $\dfrac{\partial\phi}{\partial x} = y^2 + \lambda = 0$; $\dfrac{\partial\phi}{\partial y} = 2x\cdot y + \lambda = 0$,

that, with the condition $x + y = 6$, they form a system of equations whose solutions are as follows:

$$\begin{cases} \text{For } \lambda = 0: & \Rightarrow \quad x = 6,\ y = 0. \\ \text{For } \lambda = -16: & \Rightarrow \quad x = 2,\ y = 4. \end{cases}$$

- *Sufficient or second degree condition*:

The second differential of ϕ is: $d^2\phi = 2y\cdot dx\cdot dy + 2x\cdot dy^2$,

and as from the condition equation it follows that: $dx + dy = 0$, $dy = -dx$ is also obtained, and we have that: $d^2\phi = (-2y + 2x)\,dy^2$. So:

$$\begin{cases} \text{For } x = 6,\ y = 0\text{: } d^2\phi = 12dy^2 > 0\text{; then in } (6, 0, 0) \text{ there is a local minimum.} \\ \text{For } x = 2,\ y = 4\text{: } d^2\phi = -4dy^2 < 0\text{; then in } (2, 4, 32) \text{ there is a local maximum.} \end{cases}$$

b) By reduction of variables the same conclusions will be reached, since from the conditioning equation we will have: $y = 6 - x$; and substituting this value in the objective function will result:

$$z = x(6 - x)^2 = x(36 + x^2 - 12x) = x^3 - 12x^2 + 36x\ ;$$

- *Necessary or first degree condition*:

$$z'_x = 3x^2 - 24x + 36 = 0\ ;\ \ x^2 - 8x + 12 = 0\ ;$$

$$x = \frac{8 \pm \sqrt{64 - 48}}{2} = \frac{8 \pm 4}{2} = \left\langle \begin{matrix} 6 = x_1 \\ 2 = x_2 \end{matrix} \right. .$$

There are, therefore, 2 critical points: $\left\langle \begin{matrix} P_1(6,0) \\ P_2(2,4) \end{matrix} \right. .$

- *Sufficient or second degree condition*:

$z''_{x^2} = 6x - 24$, $\rightarrow$ in P_1 is $12 > 0 \Rightarrow$ MINIMUM in P_1 (6, 0, 0)

$\rightarrow$ in P_2 is $-12 < 0 \Rightarrow$ MAXIMUM in P_2 (2, 4, 32)

Example 3

Obtain the extremes of the function $z = x^2 + y^2$, with the following condition: $x + y - 2 = 0$, applying various procedures.

a) Applying the Jacobian method, one begins by solving the system:

$$\begin{cases} J = \dfrac{\partial(f,g)}{\partial(x,y)} = 0 \\ g(x,y) = 0 \end{cases}$$, which in the present case turns out to be:

$$\begin{cases} J = \begin{vmatrix} 2x & 2y \\ 1 & 1 \end{vmatrix} = 2x - 2y = 0 \\ g = x + y - 2 = 0 \end{cases}$$

which, resolved, provides the values: $x = 1$, $y = 1$.

To determine if it is a maximum or a minimum, it is calculated:

$$\frac{\partial(J,g)}{\partial(x,y)} = \begin{vmatrix} 2 & -2 \\ 1 & 1 \end{vmatrix} = 4 > 0 \, ,$$

therefore, a local minimum resulting in x = 1, y = 1, z = 1 + 1 = 2.

b) Applying, now, the method of the Lagrange multipliers, we will begin by forming the following Lagrange or auxiliary function:

$$\phi = x^2 + y^2 + \lambda(x + y - 2).$$

- *Necessary or first degree condition*:

By canceling its two partial derivatives, we will have:

$$\begin{cases} \dfrac{\partial\phi}{\partial x} = 2x + \lambda = 0 \\ \dfrac{\partial\phi}{\partial y} = 2y + \lambda = 0 \end{cases}$$

whence x = y, which with x + y - 2 = 0 provides x = 1 and y = 1.

- *Sufficient or second degree condition*:

To determine if this solution corresponds to a maximum or a minimum, $d^2\phi$ is obtained, and depending on whether it is: $d^2\phi > 0$ or $d^2\phi < 0$, it will be a minimum or a maximum, respectively. In our case it happens that:

$$d^2\phi = 2 \cdot dx^2 + 2 \cdot dy^2 \, ,$$

and as of the condition: dx + dy = 0 we have: $d^2\phi = 4 \bullet dx^2$, which in all cases is positive, then it is a local minimum, whose value is z = 2.

c) The problem posed can also be solved directly by reducing variables, leaving us with a simple objective function of a single independent variable, since:

$$z = x^2 + y^2 \text{ ; si: } x + y - 2 = 0 \text{ ; } y = 2 - x \text{ ;}$$

$$z = x^2 + (2 - x)^2 = x^2 + 4 + x^2 - 4x = 2x^2 - 4x + 4 \, .$$

- *Necessary or first degree condition*:

$$z'_x = 4x - 4 = 0 \Rightarrow x = 1 \Rightarrow y = 2 - 1 = 1 \text{ ; } z = 1 + 1 = 2.$$

- *Sufficient or second degree condition*:

$$z''_{x^2} = 4 > 0 \Rightarrow \text{Then it is a MINIMUM, in } P_0(1, 1, 2).$$

Example 4

Using various procedures, determine the extremes of the following function: $z = x^2 + y^2$, with the condition $x^2 + 8x\cdot y + 7y^2 - 225 = 0$.

a) The Jacobian method. Let's calculate the Jacobian of the functions f and g, like this:

$$J = \frac{\partial(f,g)}{\partial(x,y)} = \begin{vmatrix} 2x & 2y \\ 2x + 8y & 8x + 14y \end{vmatrix} = 8(2x^2 + 3x\cdot y - 2y^2)\,.$$

Let's solve the system J = 0, g = 0:

$$\begin{cases} 2x^2 + 3x\cdot y - 2y^2 = 0 \\ x^2 + 8x\cdot y + 7y^2 - 225 = 0 \end{cases}$$

Multiplying the second equation by -2 and adding, we get:

$$13xy + 16y^2 = 450, \text{ from where: } x = \frac{450 - 16y^2}{13y} \quad [\text{III}]$$

and substituting this value in any of the equations, we obtain:

$$y^4 + 25y^2 - 900 = 0,$$

which is a biquadrate equation, which provides: $y^2 = \frac{-25 \pm 65}{2} = \left\langle \begin{matrix} 20 \\ -45 \end{matrix} \right.$.

From the first solution it follows that, which substituted in [III] gives, for values of x: $x = \pm\sqrt{5}$.

No real solution is obtained from the second.

To determine whether it is a maximum or a minimum, we obtain:

$$J_1(x,y) = \frac{\partial(J,g)}{\partial(x,y)} = 8\begin{vmatrix} 4x + 3y & 3x - 4y \\ 2x + 8y & 8x + 14y \end{vmatrix} = 8(26x^2 + 64x\cdot y + 74y^2),$$

and since both J_1 $(\sqrt{5}, 2\sqrt{5})$ and J_1 $(-\sqrt{5}, -2\sqrt{5})$ are positive, at both points there are relative minimum values:

$$z = (\sqrt{5})^2 + (\sqrt{20})^2 = 5 + 20 = 25.$$

b) This same problem, solved by the variable reduction method, is established as follows:

$$[\text{OPT}]\ z = x^2 + y^2,$$

with the condition: $x^2 + 8xy + 7y^2 - 225 = 0$; then substituting in the objective function, it will result:

$$z = 225 - 8x\cdot y - 6y^2 \text{ ;}$$

- *Necessary or first degree condition*:

$$\begin{cases} z'_x = -8y = 0 \Rightarrow y = 0 \\ z'_y = -8x - 12y = 0 \Rightarrow x = 0 \end{cases}$$

- *Sufficient or second degree condition*:

$z''_{x^2} = 0$; $z''_{xy} = z''_{yx} = -8$; $z''_{y^2} = -12$; then we will form the Hessian functional determinant:

$$H(x,y) = \begin{vmatrix} 0 & -8 \\ -8 & -12 \end{vmatrix} = -64 < 0,$$

it offers a "saddle point", and the problem must be solved by other methods. In this case, then, the variable reduction method has not been effective in solving the problem.

c) We will now solve the problem by applying the Lagrange multipliers method. Forming the corresponding Lagrange function and canceling its first derivatives, we will have:

$$\phi = x^2 + y^2 + \lambda(x^2 + 8x\cdot y + 7y^2 - 225).$$

- *Necessary or first degree condition*:

$$\begin{cases} \phi'_x = 2x + 2\lambda x + 8\lambda y = 0 \\ \phi'_y = 2y + 8\lambda x + 14\lambda y = 0 \\ \phi'_\lambda = x^2 + 8xy + 7y^2 - 225 \end{cases}$$

We could eliminate λ between the first two equations, but, in this case, it is preferable to obtain the possible values of λ, imposing the compatibility condition of the previous system, that is:

$$\begin{vmatrix} 2(1+\lambda) & 8\lambda \\ 8\lambda & 2(1+7\lambda) \end{vmatrix} = 0 \text{, from which it is obtained: } 9\lambda^2 - 8\lambda - 1 = 0,$$

which provides the roots: $\lambda_1 = 1$ and $\lambda_2 = -1/9$. For $\lambda_1 = 1$, by substitution in any of the equations of the previous system, $x = -2y$ is found, a value that, when substituted in the condition equation, leads to:

$$4y^2 - 16y^2 + 7y^2 = 225, \quad -5y^2 = 225, \text{ where: } y = 3i\cdot\sqrt{5},$$

pure imaginary solution that does not provide extremes.

For λ_2 = -1/9, similarly, we find y = 2x and when substituting in the condition equation it is:

$$x^2 + 16x^2 + 28x^2 = 225, \quad 45x^2 = 225, \text{ or:}$$

$$x = \pm\sqrt{5} \quad \text{therefore} \quad y = \pm 2\sqrt{5}\,, \quad z = 25.$$

- *Sufficient or second degree condition*:

To specify if it is a maximum or minimum, we obtain:

$$d^2\phi = 2(1 + \lambda)dx^2 + 16\lambda\cdot dx\cdot dy + 2(1 + 7\lambda)dy^2\,.$$

From the condition equation, by differentiation, *dy* is obtained as a function of *dx*:

$$(2x + 8y)dx + (8x + 14y)dy = 0.$$

Substituting *dy* for its value y for y = 2x, λ = -1/9, we obtain:

$d^2\phi = \frac{25}{9}dx^2$, that being positive tells us that at both critical points there are relative minimums.

If you intend to solve it alternately by forming the relevant Hessian border, you have, from the auxiliary function:

$$\phi = x^2 + y^2 + \lambda(x^2 + 8xy + 7y^2 - 225)\,;$$

$$\phi''_{x^2} = 2 + 2\lambda\,; \quad \phi''_{xy} = 8\lambda = \phi''_{yx}\,; \quad \phi''_{x\lambda} = 2x + 8y\,;$$

$$\phi''_{y^2} = 2 + 14\lambda\,; \quad \phi''_{y\lambda} = 8x + 14y\,; \quad \text{and the Hessian determinant will result:}$$

$$H(x,y,\lambda) = \begin{vmatrix} 2+2\lambda & 8\lambda & 2x+8y \\ 8\lambda & 2+14\lambda & 8x+14y \\ 2x+8y & 8x+14y & 0 \end{vmatrix} = 8\lambda(8x + 14y)(2x + 8y) + 8\lambda(2x +$$

$$+ 8y)(8x + 14y) - (2x + 8y)^2(2 + 14\lambda) - (2 + 2\lambda)(8x + 14y)^2 =$$

$$= 16\lambda(8x + 14y)(2x + 8y) - (4x^2 + 64y^2 + 32xy)(2 + 14\lambda) - (2 + 2\lambda)(64x^2 +$$

$+ 196\, y^2 + 224\, xy) = \ldots\ldots$, with a very laborious resolution, to which the obtained values of λ, x and y will have to be substituted, making it more practical to find the numerical value of the Hessian functional determinant for both critical points obtained. And so, for $(\sqrt{5}, 2\sqrt{5})$ you will have:

$$H(\sqrt{5},2\sqrt{5},-1/9)=\begin{vmatrix}16/9 & -8/9 & 18\sqrt{5}\\ -8/9 & 4/9 & 36\sqrt{5}\\ 18\sqrt{5} & 36\sqrt{5} & 0\end{vmatrix}=-18000<0,$$ then it is a relative or local minimum. Similarly, for $(-\sqrt{5},-2\sqrt{5})$ you will have:

$$H(-\sqrt{5},-2\sqrt{5},-1/9)=\begin{vmatrix}16/9 & -8/9 & -18\sqrt{5}\\ -8/9 & 4/9 & -36\sqrt{5}\\ -18\sqrt{5} & -36\sqrt{5} & 0\end{vmatrix}=-18000<0,$$ so it will also be a relative minimum.

Example 5

Find, by various procedures, the maximums and minimums of the function: $u = x\cdot y^2\cdot z^3$, with the condition $x + y + z = 12$, where x, y, z are positive.

a) Being x, y, z positive, the ends of the function u will coincide with those of the function ln u, where by *ln* we denote the natural or Neperian logarithm. Therefore, the Lagrange or auxiliary function will be:

$$\Phi\ (x, y, z) = \ln x + 2\ln y + 3\ln z + \lambda\ (x + y + z - 12).$$

- *Necessary or first degree conditions*:

Its zero-matched partial derivatives provide the following system:

$$\begin{cases}\Phi'_x=\dfrac{1}{x}+\lambda=0\\ \Phi'_y=\dfrac{2}{y}+\lambda=0\\ \Phi'_z=\dfrac{3}{z}+\lambda=0\\ \Phi'_\lambda=x+y+z-12=0\end{cases}\quad \text{, or , } -\frac{1}{\lambda}=x=\frac{y}{2}=\frac{z}{3},$$

that with the condition $x + y + z = 12$, they provide us, in short, with the values:

$$x = 2,\quad y = 4,\quad z = 6,\quad \lambda=-\frac{1}{2}.$$

- *Sufficient or second degree conditions*:

If we now calculated the second differential of u at that point, we would find that in (2, 4, 6) there is a maximum. However, we are going to start from the relevant hessian functional determinant, with what:

$$
\left|\begin{array}{l} \Phi''_{x^2} = -\dfrac{1}{x^2} \\ \Phi''_{xy} = 0 \\ \Phi''_{xz} = 0 \\ \Phi''_{x\lambda} = 1 \end{array}\right|
\begin{array}{l} \Phi''_{y^2} = -\dfrac{2}{y^2} \\ \Phi''_{yx} = 0 \\ \Phi''_{yz} = 0 \\ \Phi''_{y\lambda} = 1 \end{array}\left|
\begin{array}{l} \Phi''_{z^2} = -\dfrac{3}{z^2} \\ \Phi''_{zx} = 0 \\ \Phi''_{zy} = 0 \\ \Phi''_{z\lambda} = 1 \end{array}\right|
\begin{array}{l} \Phi''_{\lambda^2} = 0 \\ \Phi''_{\lambda x} = 1 \\ \Phi''_{\lambda y} = 1 \\ \Phi''_{\lambda z} = 1 \end{array}\right|
$$

$$
H(x, y, z, \lambda) = \begin{vmatrix} -\dfrac{1}{x^2} & 0 & 0 & 1 \\ 0 & -\dfrac{2}{y^2} & 0 & 1 \\ 0 & 0 & -\dfrac{3}{z^2} & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} = (\text{developing by the elements of the 4th row}) =
$$

$$
= -\begin{vmatrix} 0 & 0 & 1 \\ -\dfrac{2}{x^2} & 0 & 1 \\ 0 & -\dfrac{3}{z^2} & 1 \end{vmatrix} + \begin{vmatrix} -\dfrac{1}{x^2} & 0 & 1 \\ 0 & 0 & 1 \\ 0 & -\dfrac{3}{z^2} & 1 \end{vmatrix} - \begin{vmatrix} -\dfrac{1}{x^2} & 0 & 1 \\ 0 & -\dfrac{2}{x^2} & 1 \\ 0 & 0 & 1 \end{vmatrix} =
$$

$$
= -\frac{6}{y^2\, z^2} - \frac{3}{x^2\, z^2} - \frac{2}{x^2\, y^2} < 0 \quad (3\ \text{variables}),
$$

then it could be maximum or minimum.

The same problem, solved directly (longer), offers:

- *Necessary or first degree condition*:

$$
\Phi = x \cdot y^2 \cdot z^3 + \lambda\,(x + y + z - 12);
$$

$$
\left.\begin{array}{l} \Phi'_x = y^2\, z^3 + \lambda = 0 \\ \Phi'_y = 2\, x\, y\, z^3 + \lambda = 0 \\ \Phi'_z = 3\, x\, y^2\, z^2 + \lambda = 0 \\ \Phi'_\lambda = x + y + z - 12 = 0 \end{array}\right\} \text{ from where:}
$$

$y^2\, z^3 = 2\, x\, y\, z^3 = 3\, x\, y^2\, z^2$; now divided by ($y\, z^2$) is: $y\, z = 2\, x\, z = 3\, x\, y$; $\left.\begin{array}{l} y = 2x \\ z = 3x \end{array}\right\}$

So: $x + 2\,x + 3\,x = 12 \left\langle \begin{array}{l} x = 2 \\ y = 4 \\ z = 6 \end{array}\right.$, then at the critical point P(2, 4, 6) there is a relative or local extreme.

- *Sufficient or second degree condition*:

$$\left|\begin{array}{l} \Phi''_{x^2}=0 \\ \Phi''_{xy}=2\,y\,z^3 \\ \Phi''_{xz}=3\,y^2\,z^2 \\ \Phi''_{x\lambda}=1 \end{array}\right. \left|\begin{array}{l} \Phi''_{y^2}=2\,x\,z^3 \\ \Phi''_{yx}=2\,y\,z^3 \\ \Phi''_{yz}=6\,x\,y\,z^2 \\ \Phi''_{y\lambda}=1 \end{array}\right. \left|\begin{array}{l} \Phi^2_{z^2}=6\,x\,y^2\,z \\ \Phi''_{zx}=3\,y^2\,z^2 \\ \Phi''_{zy}=6\,x\,y\,z^2 \\ \Phi''_{z\lambda}=1 \end{array}\right. \left|\begin{array}{l} \Phi''_{\lambda^2}=0 \\ \Phi''_{\lambda x}=1 \\ \Phi''_{\lambda y}=1 \\ \Phi''_{\lambda z}=1 \end{array}\right|$$

, with which:

$$H(x,y,z,\lambda)=\begin{vmatrix} 0 & 2\,y\,z^3 & 3\,y^2\,z^2 & 1 \\ 2\,y\,z^3 & 2\,x\,z^3 & 6\,x\,y\,z^2 & 1 \\ 3\,x^2\,z^2 & 6\,x\,y\,z^2 & 6\,x\,y^2\,z & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}=$$

(developing by the elements of the 4th row) =

$$=-\begin{vmatrix} 2\,y\,z^3 & 3\,y^2\,z^2 & 1 \\ 2\,x\,z^3 & 6\,x\,y\,z^2 & 1 \\ 6\,x\,y\,z^2 & 6\,x\,y^2\,z & 1 \end{vmatrix}+\begin{vmatrix} 0 & 3\,y^2\,z^2 & 1 \\ 2\,y\,z^3 & 6\,x\,y\,z^2 & 1 \\ 3\,y^2\,z^2 & 6\,x\,y^2\,z & 1 \end{vmatrix}-\begin{vmatrix} 0 & 2\,y\,z^3 & 1 \\ 2\,y\,z^3 & 2\,x\,z^3 & 1 \\ 3\,y^2\,z^2 & 6\,x\,y\,z^2 & 1 \end{vmatrix}=$$

= ……………….. (reaching the same conclusions as through the previous simplified process).

With $\left\langle\begin{array}{l} x=2 \\ y=4 \\ z=6 \end{array}\right.$, substituting these values in the previous Hessian, we have to:

$$H(x,y,z,\lambda)=\begin{vmatrix} 0 & 1728 & 1728 & 1 \\ 1728 & 864 & 1728 & 1 \\ 1728 & 1728 & 1152 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}=-2\,985\,984<0\,,$$

then it can be maximum or minimum (3 variables), and we still cannot find the definitive solution.

b) Solving it through a simple example (by trial and error), we would have the following:

$$\left.\begin{array}{l} x=1 \\ y=5 \\ \underline{z=6} \\ \sum=12 \end{array}\right\} u=x\cdot y^2\cdot z^3=1\times 25\times 216=5400$$

, and alternatively:

$$\left.\begin{array}{l} x=2 \\ y=4 \\ \underline{z=6} \\ \sum=12 \end{array}\right\} u = x \cdot y^2 \cdot z^3 = 2\times 16\times 216 = 6912,$$ and so it will be for all possible combinations of three positive quantities that meet the restriction imposed in the statement, then it seems obvious that at the critical point $P_0(2, 4, 6)$ there is a RELATIVE MAXIMUM.

c) Next, we will try the variable reduction method, and we will have to:

x = 12 - y - z, with what substituting in the objective function, we will have:

$\Phi = (12 - y - z) \cdot y^2 \cdot z^3 = 12\, y^2 z^3 - y^3 z^3 - y^2 \cdot z^4 = \Phi\ (y, z)$, which is already a case of unconditional extremes and only 2 variables.

- *Necessary or first degree condition*:

$$\left.\begin{cases} \Phi'_y = 24\, z^3 \cdot y - 3\, y^2\, z^3 - 2\, y\, z^4 = 0 \\ \Phi'_z = 36\, y^2 \cdot z^2 - 3\, y^3\, z^2 - 4\, y^2\, z^3 = 0 \end{cases}\right\} \text{ from where:}$$

$$\begin{array}{l} 24 - 3y - 2z = 0 \\ \underline{-36 + 3y + 4z = 0} \\ -12 + 2z = 0 \Rightarrow z = 6;\ \ y = \dfrac{24 - 2\,z}{3} = \dfrac{24 - 12}{3} = 4;\ \ x = 12 - 4 - 6 = 2. \end{array}$$

- *Sufficient or second degree condition*:

We will form the Hessian determinant:

$$H(y,z) = \begin{vmatrix} \Phi''_{y^2} & \Phi''_{yz} \\ \Phi''_{yz} & \Phi''_{z^2} \end{vmatrix}, \text{ with the following real values:}$$

$$\begin{cases} \Phi''_{y^2} = 24\, z^3 - 6\, y\, z^3 - 2\, z^4 = 5184 - 5184 - 2592 = -2592 \\ \Phi''_{yz} = 72\, y\, z^2 - 9\, y^2\, z^2 - 8\, y\, z^3 = 10368 - 5184 - 6912 = -1728 \\ \Phi''_{z^2} = 72\, y^2\, z - 6\, y^3\, z - 12\, y^2\, z^2 = 6912 - 2304 - 6912 = -2304 \end{cases}$$

$$H(4,6) = \begin{vmatrix} -2592 & -1728 \\ -1728 & -2304 \end{vmatrix} = 5\,971\,968 - 2\,985\,984 = 2\,985\,984 > 0,$$

with $\Phi''_{y^2} = -2592 < 0$,

then it is a relative or local MAXIMUM at the critical point $P_0(2, 4, 6)$, with a value: $u = x \cdot y^2 \cdot z^3 = 2 \times 16 \times 216 = 6912$.

Note the greater ease of resolution obtained using this last procedure (reduction or elimination of variables) in the present example compared to the Lagrange multipliers method, as well as the fact that it has allowed us to easily discriminate the nature of the critical point found as a consequence of the application of the necessary or first degree condition. Hence the interest in their employment in most cases that occur in practice.

Example 6

Find, by various procedures, the maximum of the product: x·y·z, when x + y + z = a; where x, y, z are positive.

a) We will form, in principle, the following auxiliary or Lagrangian function:

$$\Phi = x \cdot y \cdot z + \lambda(x + y + z - a) .$$

- *Necessary or first degree condition*:

$$\left.\begin{aligned}
\Phi'_x &= \frac{\partial \Phi}{\partial x} = y \cdot z + \lambda = 0 \\
\Phi'_y &= \frac{\partial \Phi}{\partial y} = x \cdot z + \lambda = 0 \\
\Phi'_z &= \frac{\partial \Phi}{\partial z} = x \cdot y + \lambda = 0 \\
\Phi'_\lambda &= \frac{\partial \Phi}{\partial \lambda} = x + y + z - a = 0
\end{aligned}\right\} x = y = z;\ 3x = a;\ \text{from where:}$$

$$x = \frac{a}{3};\quad y = \frac{a}{3};\quad z = \frac{a}{3};\ P_{\text{máx.}} = x \cdot y \cdot z = \frac{a^3}{27};\ \text{with: } \lambda = -\frac{a^2}{9} .$$

- *Sufficient or second degree condition*:

$$\begin{vmatrix}
\Phi''_{x^2} = 0 & \Phi''_{y^2} = 0 & \Phi''_{z^2} = 0 & \Phi''_{\lambda^2} = 0 \\
\Phi''_{xy} = z & \Phi''_{yx} = z & \Phi''_{zx} = y & \Phi''_{\lambda x} = 1 \\
\Phi''_{xz} = y & \Phi''_{yz} = x & \Phi''_{zy} = x & \Phi''_{\lambda y} = 1 \\
\Phi''_{x\lambda} = 1 & \Phi''_{y\lambda} = 1 & \Phi''_{z\lambda} = 1 & \Phi''_{\lambda z} = 1
\end{vmatrix};\ \text{the relevant Hessian border, will be:}$$

$$H(x,y,z,\lambda) = \begin{vmatrix} 0 & z & y & 1 \\ z & 0 & x & 1 \\ y & x & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} = -\begin{vmatrix} z & y & 1 \\ 0 & x & 1 \\ x & 0 & 1 \end{vmatrix} + \begin{vmatrix} 0 & y & 1 \\ z & x & 1 \\ y & 0 & 1 \end{vmatrix} - \begin{vmatrix} 0 & z & 1 \\ z & 0 & 1 \\ y & x & 1 \end{vmatrix} =$$

= - (x z + x y - x^2) + (y^2 – x y – z y) – (z y + z x – z^2) = (operating properly) = - 2 x^2 – 2 x^2 – 2 x^2 + x^2 + x^2 + x^2 = - 6 x^2 + 3 x^2 = - 3 x^2 = $-\frac{a^2}{3} < 0$ (3 variables), then it could be maximum or minimum, and you will have to try to solve this problem by some other procedure.

b) Variable reduction method:

By making the substitution on the objective function: z = a - x - y, the following 2-variable function results:

$$\Phi(x,y) = x\cdot y(a - x - y) = a\cdot x\cdot y - x^2\cdot y - x\cdot y^2;$$

- Necessary or first degree condition:

$$\left.\begin{array}{l}\Phi'_x = a\,y - 2\,y\,x - y^2 = 0\\ \Phi'_y = a\,x - x^2 - 2\,y\,x = 0\end{array}\right\} \text{ from which it results } \left.\begin{array}{l}a - 2\,x - y = 0\\ a - x - 2\,y = 0\end{array}\right\}$$

and, in short, $x = y = z = \dfrac{a}{3}$ (critical point).

- Sufficient or second degree condition:

$$\left\{\begin{array}{l}\Phi''_{x^2} = -2\,y\\ \Phi''_{xy} = \Phi''_{xy} = a - 2\,x - 2\,y\\ \Phi''_{y^2} = -2\,x\end{array}\right., \text{ and so: } H(x,y,z) = \begin{vmatrix} -2\,y & a - 2(x+y)\\ a - 2(x+y) & -2\,x\end{vmatrix} =$$

$$= -a^2 - 4\,x^2 - 4\,y^2 - 4\,x\,y + 4\,a\,x + 4\,a\,y = -a^2 - \frac{4\,a^2}{3} + \frac{8\,a^2}{3} = \frac{4\,a^2}{3} - a^2 = \frac{a^2}{3} > 0\,;$$

$\Phi''_{x^2} = -2y = \dfrac{-2\,a}{3} < 0$, then there is a MAXIMUM RELATIVE at the point $P(a/3, a/3, a/3)$, which effectively solves the problem posed.

Thus, as in the previous example, the variable reduction or substitution method has made it possible to easily discriminate the nature of the critical point found, a circumstance that had not been achieved by the application of the Lagrange multipliers method.

Example 7

Let's consider the following program:

$$\begin{array}{ll}\text{Optimize} & f(x_1,x_2) = x_1^2 + 2x_2 + 2\\ \text{subject to:} & x_1^3 - x_2 - 2 = 0.\end{array}$$

Solution:

a) From the given restriction we can obtain: $x_2 = x_1^3 - 2$, and substituting in the objective function (variable elimination method) we have:

$\Phi(x_1) = x_1^2 + 2x_1^3 - 2$, which is already a real function of a single real variable.

Now let's find the ends of this function:

$$\frac{d\Phi}{dx_1} = 2x_1 + 6x_1^2 = 0 \rightarrow \begin{cases} x_1 = 0 \\ x_2 = -1/3 \end{cases}$$

The second drift will be:

$$\frac{d^2\Phi}{dx_1^2} = 2 + 12x_1, \quad \frac{d^2\Phi(0)}{dx_1^2} = 2 > 0 \quad \text{and} \quad \frac{d^2\Phi(-1/3)}{dx_1^2} = -2 < 0 .$$

Therefore, at these points, the objective function Φ has a relative minimum and maximum, respectively.

Substituting these points in $x_2 = x_1^3 - 2$ we can conclude that (0, -2) and (-1/3, -55/27) are, respectively, the relative minimum and maximum of the original program (Balbás and Gil, 2004).

b) The problem can also be solved by formulating the corresponding Lagrangian function: $\Phi = x_1^2 + 2x_2 + 2 + \lambda(x_1^3 - x_2 - 2)$.

- *Necessary or first degree condition*:

$$\left.\begin{array}{l} \Phi'_{x_1} = 2x_1 + 3 \cdot \lambda \cdot x_1^2 = 0 \\ \Phi'_{x_2} = 2 - \lambda = 0 \\ \Phi'_{\lambda} = x_1^3 - x_2 - 2 = 0 \end{array}\right\}$$

De la resolución de este sistema, con λ = 2, surgen los dos puntos críticos: (0, -2) y $\left(-1/3, -55/27\right)$.

From the resolution of this system, with λ = 2, the two critical points arise: (0, -2) and $\left(-1/3, -55/27\right)$.

- The critical point (0, -2) offers a value of the objective function:

$$\Phi = 2 \times (-2) + 2 = -2 \text{ (MINIMUM RELATIVE).}$$

- The critical point $\left(-1/3, -55/27\right)$ offers a value of the objective function:

$$\Phi = \frac{3}{27} - \frac{110}{27} + \frac{54}{27} = -\frac{53}{27} \approx -1.96 > -2 \text{ (MAXIMUM RELATIVE).}$$

- *Sufficient or second grade condition*:

To corroborate the above, let's form the corresponding relevant Hessian border, that is:

$\Phi''_{x_1^2} = 2 + 12x_1$; $\Phi''_{x_2^2} = 0$; $\Phi''_{x_1x_2} = 0$; $\Phi''_{x_2\lambda} = -1$; $\Phi''_{x_1\lambda} = 3x_1^2$; and so:

$$H(x_1, x_2, \lambda) = \begin{vmatrix} \Phi''_{x_1^2} & \Phi''_{x_1x_2} & \Phi''_{x_1\lambda} \\ \Phi''_{x_1x_2} & \Phi''_{x_2^2} & \Phi''_{x_2\lambda} \\ \Phi''_{x_1\lambda} & \Phi''_{x_2\lambda} & \Phi''_{\lambda^2} \end{vmatrix} = \begin{vmatrix} 2+12x_1 & 0 & 3x_1^2 \\ 0 & 0 & -1 \\ 3x_1^2 & -1 & 0 \end{vmatrix} = -2 - 12x_1 \Rightarrow$$

$$\Rightarrow \begin{cases} \text{con } x_1 = 0 \Rightarrow H = -2 < 0 \Rightarrow \text{LOCAL MINIMUM} \\ \text{con } x_1 = -\frac{1}{3} \Rightarrow H = -2 + 4 = 2 > 0 \Rightarrow \text{LOCAL MAXIMUM} \end{cases}$$

which offers the same result as that which was deduced directly from the application of the necessary condition.

CASE STUDY

At this point, let us see that identical formulations to those proposed by this author in his studies for the case of free conductions (see Chap. I, epigraphs 4.3. and following of the book "Five subjects of Hydrology and Hydraulics", Universitat Internacional de Catalunya, Tortosa, 2003, cited in the bibliography) can be applied, with the corresponding corrections, in the calculation and design of forced or pressure pipes. For this, the formulas corresponding to the 6 categories of roughness have been used, and they are expressed below in the following table, depending on the material of the tube and for pipes used or in service.

These formulas will adopt the general configuration: $V = K \cdot R^{\beta} \cdot J^{0'5}$, in which the speed V(m/s) is given as a function of the hydraulic radius R(m), of the loss of unit load J(m/m) and of coefficients according to the various categories of roughness. Namely:

Table 1. Coefficients of the formulation proposed by Franquet according to the different categories of roughness.

Roughness degree (k)	Material	K	β
1	Plastics, glass, brass, stretched copper	86.85	0.62150
2	Fiber cement, aluminum	78.29	0.63455
3	Steel, other metals	70.02	0.64760
4	Uncoated cast iron	63.92	0.65560
5	Uncoated concrete	56.24	0.66540
6	Uncoated ceramic	49.51	0.67725

Source: Franquet, 2005.

The previous formulation, however, is more practical to apply depending on the internal diameter D(m) of the pipe and the flow Q(m^3/s) circulating through it, so, for the basic case studied (pipe in used), we would have, correlatively, the following expressions, in which the unit head loss J(m/m) has also been solved and the intermediates obtained by linear interpolation have been included:

Table 2. Proposed expressions of speed, flow and unit pressure drop for pipes in service.

Roughness (k)	**V (m/s)**	**Q (m^3/s)**	**J (m/m)**
1.0	$\mathbf{36.69 \cdot D^{0.6215} \cdot J^{0.5}}$	$\mathbf{28.82 \cdot D^{2.6215} \cdot J^{0.5}}$	$\mathbf{0.000743 \cdot V^{2} \cdot D^{-1.243}}$
1.5	$34.59 \cdot D^{0.62802} \cdot J^{0.5}$	$27.16 \cdot D^{2.62802} \cdot J^{0.5}$	$0.000845 \cdot V^{2} \cdot D^{-1.256}$
2.0	$\mathbf{32.48 \cdot D^{0.63455} \cdot J^{0.5}}$	$\mathbf{25.51 \cdot D^{2.63455} \cdot J^{0.5}}$	$\mathbf{0.000948 \cdot V^{2} \cdot D^{-1.2691}}$
2.5	$30.51 \cdot D^{0.6411} \cdot J^{0.5}$	$23.96 \cdot D^{2.6411} \cdot J^{0.5}$	$0.001088 \cdot V^{2} \cdot D^{-1.2821}$
3.0	$\mathbf{28.53 \cdot D^{0.6476} \cdot J^{0.5}}$	$\mathbf{22.41 \cdot D^{2.6476} \cdot J^{0.5}}$	$\mathbf{0.001229 \cdot V^{2} \cdot D^{-1.2952}}$
3.5	$27.14 \cdot D^{0.6516} \cdot J^{0.5}$	$21.32 \cdot D^{2.6516} \cdot J^{0.5}$	$0.001368 \cdot V^{2} \cdot D^{-1.3032}$
4.0	$\mathbf{25.76 \cdot D^{0.6556} \cdot J^{0.5}}$	$\mathbf{20.23 \cdot D^{2.6556} \cdot J^{0.5}}$	$\mathbf{0.001507 \cdot V^{2} \cdot D^{-1.3112}}$
4.5	$24.06 \cdot D^{0.6605} \cdot J^{0.5}$	$18.89 \cdot D^{2.6605} \cdot J^{0.5}$	$0.001753 \cdot V^{2} \cdot D^{-1.321}$
5.0	$\mathbf{22.36 \cdot D^{0.6654} \cdot J^{0.5}}$	$\mathbf{17.56 \cdot D^{2.6654} \cdot J^{0.5}}$	$\mathbf{0.002 \cdot V^{2} \cdot D^{-1.3308}}$
5.5	$20.86 \cdot D^{0.6713} \cdot J^{0.5}$	$16.38 \cdot D^{2.6713} \cdot J^{0.5}$	$0.002334 \cdot V^{2} \cdot D^{-1.3426}$
6.0	$\mathbf{19.36 \cdot D^{0.67725} \cdot J^{0.5}}$	$\mathbf{15.21 \cdot D^{2.67725} \cdot J^{0.5}}$	$\mathbf{0.002668 \cdot V^{2} \cdot D^{-1.3545}}$

Source: self made.

A more complete and justified view of the minimization of unit head losses for each of the six (or eleven) roughness categories previously defined, can be carried out from the conditioning equation: $V = 1.4466 \times D + 0.638$, also deduced by the undersigned, which represents the maximum admissible speed of water circulation by the pipeline as a function of its internal diameter (Franquet, 2005).

If we consider, v. gr., the roughness category of a reinforced concrete pipe (k = 5), will have the corresponding proposed expression of the loss of unit load that we will try to minimize by the method of the Lagrange operators applying the necessary or first degree condition, so:

$$V = K \cdot R^{\beta} \cdot J^{0'5} = 56.24 \times \left(\frac{D}{4}\right)^{0.6654} \times J^{0.5} = 22.36 \times D^{0.6654} \times J^{0.5},$$

and also in view of Table 2:

$$\begin{cases} J = 0.002 \times V^{2} \times D^{-1.3308} \\ J'_{V} = 0.004 \times V \times D^{-1.3308} = 0 \\ J'_{D} = 0.002 \times V^{2} \times (-1.3308) \times D^{-2.3308} = 0 \end{cases}$$

Being necessarily V and D positive, the conditioned ends of the function J will coincide with those of the logarithmic function ln J. This is done for ease of calculation, since in this way the product becomes a sum or addition. Therefore, we form the following Lagrangian or auxiliary function:

$$\Phi\ (V, D) = \ln 0.002 + 2 \times \ln V - 1.3308 \times \ln D + \lambda\ (V - 1.4466 \cdot D - 0.638).$$

The *necessary or first degree condition* requires that:

$$\left.\begin{aligned} \Phi'_V &= \frac{2}{V} + \lambda = 0 \\ \Phi'_D &= -\frac{1.3308}{D} - 1.4466 \cdot \lambda = 0 \\ \Phi'_\lambda &= V - 1.4466 \cdot D - 0.638 = 0 \end{aligned}\right\} \text{ whence it follows: } \lambda = -\frac{2}{V} = -\frac{1.3308}{1.4466 \cdot D}\ ;$$

$$\frac{2}{1.4466 \cdot D + 0.638} = \frac{1.3308}{1.4466 \cdot D}\ ;\ 2.8932 \cdot D = 1.9251 \cdot D + 0.849;$$

$0.9681 \times D = 0.849 \quad \Rightarrow \quad D = 0.877$ m, which limits the advisable maximum speed, as we have seen, to:

$$V = 1.4466 \times 0.877 + 0.638 = 1.91 \text{ m/s ; and so:}$$

$\lambda = -\frac{2}{V} = -\frac{2}{1.91} = -1.049$; with a flow of: $Q = 0.7854 \times V \times D^2 =$
$= 0.7854 \times 1.91 \times 0.877^2 = 1.15$ m^3/s, and a unit pressure drop of:

$$J = \frac{0.002 \times 1.91^2}{0.877^{1.3308}} = 0.00866 \text{ m/m.}$$

Second grade or sufficient condition:

For this, we will form the relevant Hessian bordering functional determinant, with which:

$$H(V,D,\lambda) = \begin{vmatrix} \Phi''_{V^2} & \Phi''_{VD} & \Phi''_{V\lambda} \\ \Phi''_{VD} & \Phi''_{D^2} & \Phi''_{D\lambda} \\ \Phi''_{V\lambda} & \Phi''_{D\lambda} & \Phi''_{\lambda^2} \end{vmatrix} = \begin{vmatrix} -\frac{2}{V^2} & 0 & 1 \\ 0 & \frac{1.3308}{D^2} & -1.4466 \\ 1 & -1.4466 & 0 \end{vmatrix} = -\frac{1.3308}{D^2} + \frac{4.1853}{V^2} =$$

$= -\frac{1.3308}{0.877^2} + \frac{4.1853}{1.91^2} = 1.147 - 1.730 = -0.58 < 0 \rightarrow$ then it is a relative or local MINIMUM at the critical point $P_0(1.91, 0.877, 0.00866)$.

Trial testing:

- If we assume: $D = 1.000$ m $\Rightarrow V = 1.4466 \times 1 + 0.638 = 2.08$ m/s.

$$\text{So: } J = \frac{0.002 \times 2.08^2}{1^{1.3308}} = 0.00869\, m/m.$$

- If we assume: D = 0.500 m $\Rightarrow$ V= 1.4466 $\times$ 0.5 + 0.638 = 1.36 m/s.

$$\text{So: } J = \frac{0.002 \times 1.36^2}{0.5^{1.3308}} = 0.00931\ m/m.$$

- If we assume: D = 2.000 m $\Rightarrow$ V = 1.4466 $\times$ 2 + 0.638 = 3.53 m/s.

$$\text{So: } J = \frac{0.002 \times 3.53^2}{2^{1.3308}} = 0.00991\ m/m.$$

- If we assume: D = 0.100 m $\Rightarrow$ V = 1.4466 $\times$ 0.1 + 0.638 = 0.78 m/s.

$$\text{So: } J = \frac{0.002 \times 0.78^2}{0.1^{1.3308}} = 0.02624\ m/m.$$

The following table shows -as a summary of the previous determination- the presence of the minimum of the objective function in an environment sufficiently representative of it. So:

D (m)	**J (m/m)**
0.100	0.02624
0.500	0.00931
0.877	**0.00866**
1.000	0.00869
2.000	0.00991

It follows, therefore, that complying with the law of average speed as a function of the internal diameter that we propose here, and specifically for the roughness category k = 5 (uncoated concrete), the minimum value of the unit load loss is J = 0.00866 m/m, which takes place for V = 1.91 m/s, D = 877 mm and Q = 1.15 m^3/s.

These same determinations can be made for the remaining five categories of roughness, ultimately resulting in the following table:

Table 3. Minimum value of J for the different categories of roughness.

Roughness (k)	**J (m/m)**	**D (m)**	**V (m/s)**	**Q (m^3/s)**
1	0.00315	0.724	1.69	0.70
2	0.00405	0.766	1.75	0.80
3	0.00529	0.810	1.81	0.93
4	0.00650	0.840	1.85	1.03
5	0.00866	0.877	1.91	1.15
6	0.01159	0.925	1.98	1.33

Likewise, for each of the different categories of roughness, we can see, in the following graphs, the greater detail of the absolute minimums of the function J (D). Namely:

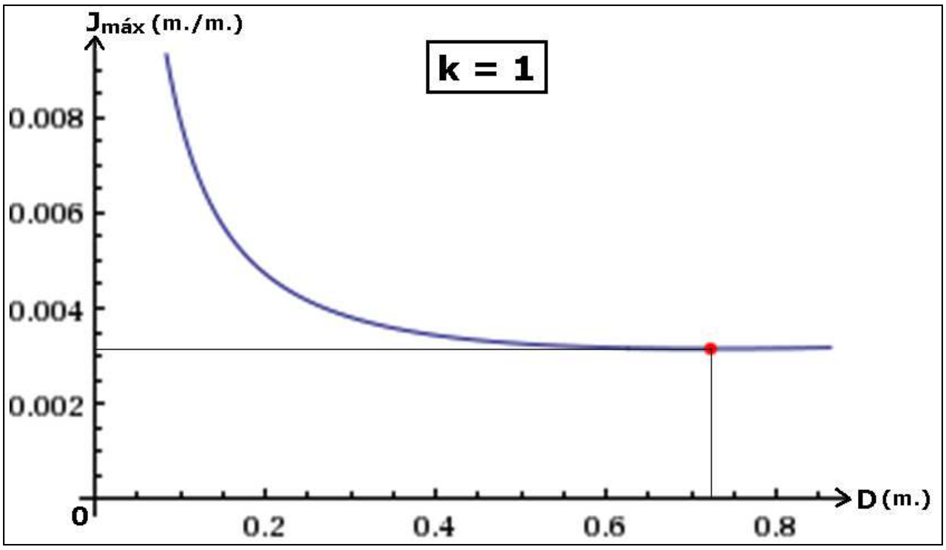

Fig. 1. Absolute minimum of J (D) for k = 1.

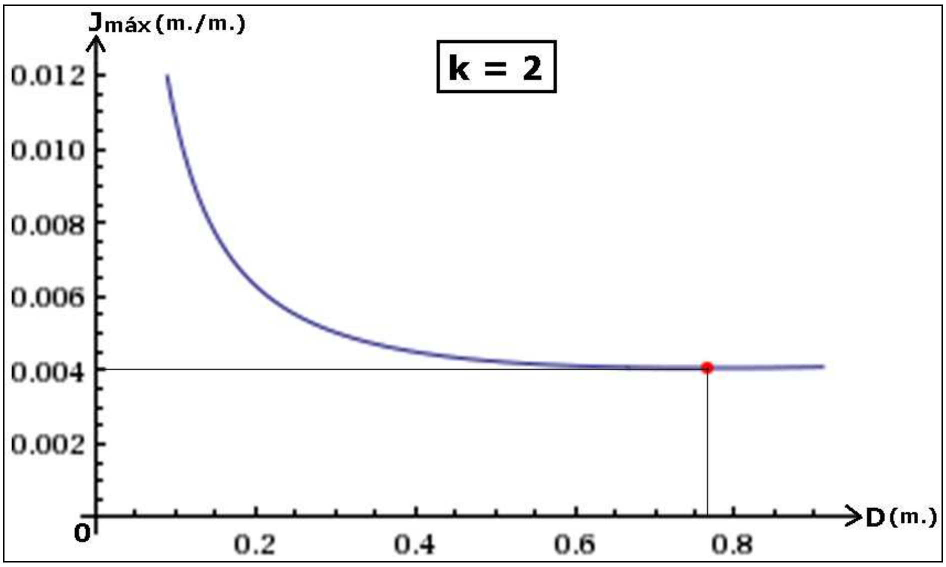

Fig. 2. Absolute minimum of J (D) for k = 2.

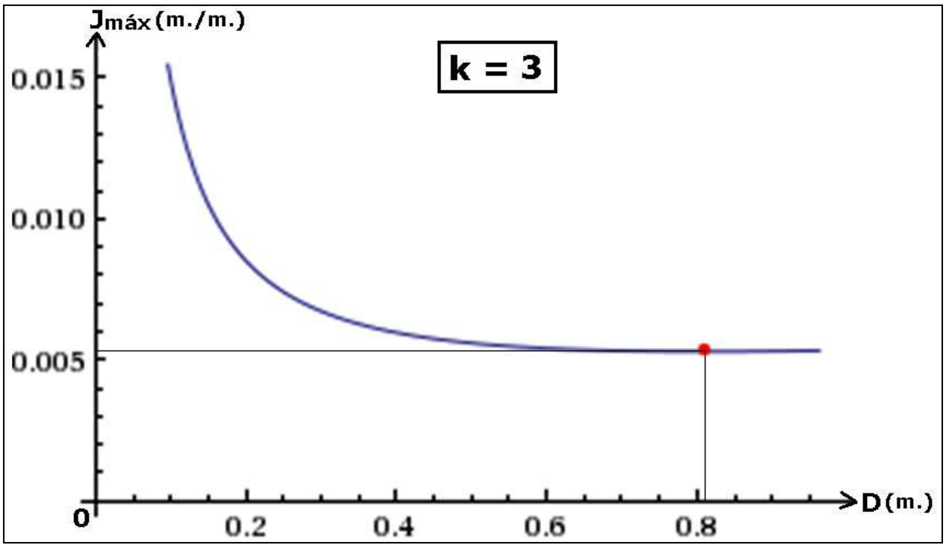

Fig. 3. Absolute minimum of J (D) for k = 3.

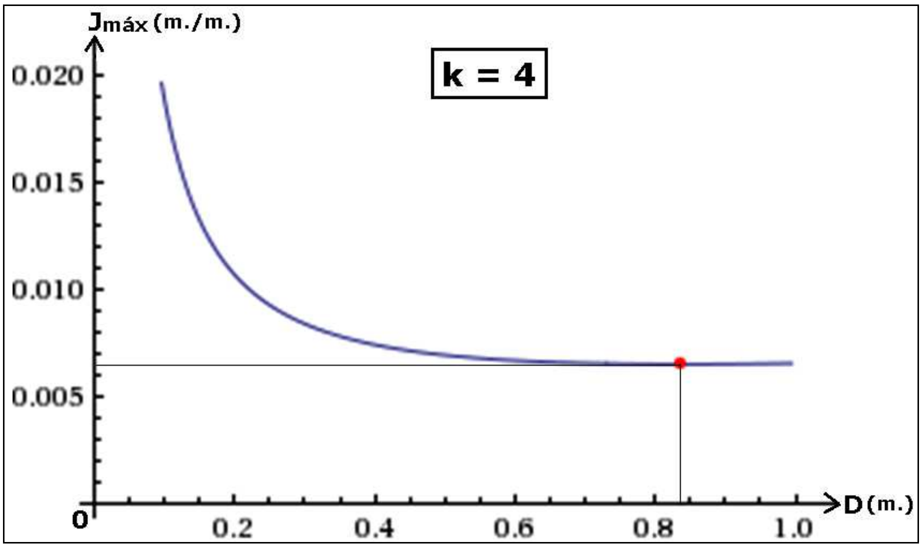

Fig. 4. Absolute minimum of J (D) for k = 4.

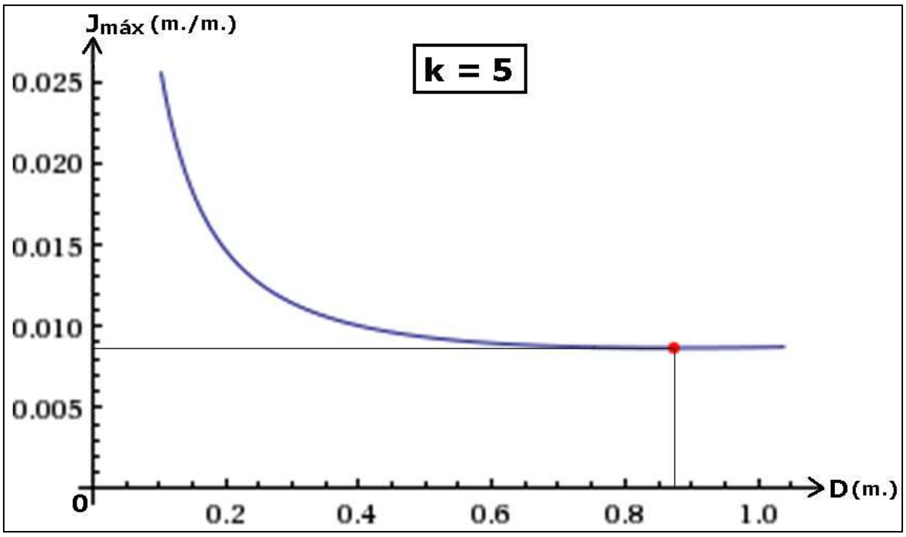

Fig. 5. Absolute minimum of J (D) for k = 5.

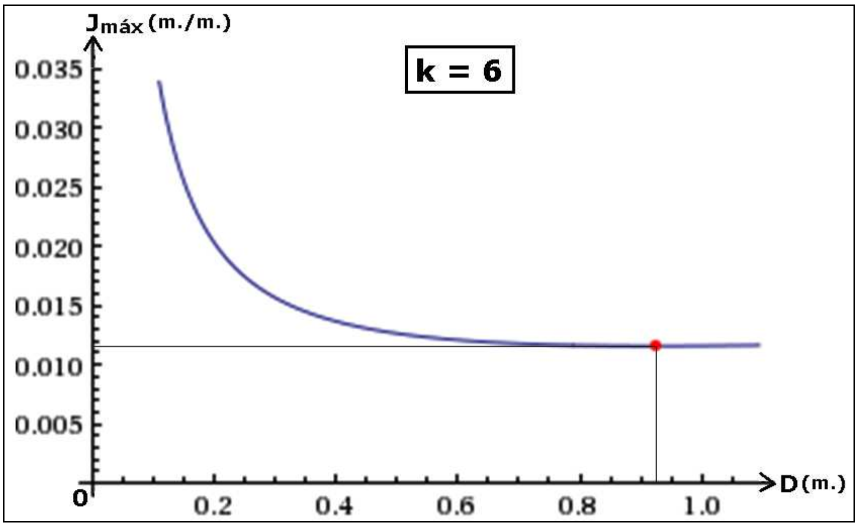

Fig. 6. Absolute minimum of J (D) for k = 6.

Verification by reduction of variables:

The new objective function to be optimized, with a single variable, will be given by substituting in it the conditioning equation, that is:

$J = 0.002 \times (1.446 \times D + 0.638)^2 \times D^{-1.3308}$; and developing:

$J = 0.002\ (2.0927 \times D^2 + 0.407 + 1.8459 \times D) \times D^{-1.3308} =$

$= 0.002\ (2.0927 \times D^{0.6692} + 0.407 \times D^{-1.3308} + 1.8459 \times D^{-0.3308})$;

and the necessary or first degree condition will require that:

$$J'_D = 0.002\left(1.4004 \times D^{-0.3308} - 0.5416 \times D^{-2.3308} - 0.6106 \times D^{-1.3308}\right) = 0\,,$$

and operating properly: $\frac{1.4004}{D^{0.3308}} - \frac{0.5416}{D^{2.3308}} - \frac{0.6106}{D^{1.3308}} = 0$; from where:

$1.4004 \times D^2 - 0.5416 - 0.6106 \times D = 0$; then the only positive root will be:

$$D = \frac{0.6106 + \sqrt{0.3729 + 3.0338}}{2.8008} = \frac{0.6106 + 1.8457}{2.8008} = 0.877\,\text{m}\,,$$

although with a shorter and simpler resolution process.

The sufficient or second degree condition corroborates the presence of a relative minimum, as can be verified from the 2nd derivative:

$$J''_D = 0.002\left(-0.4633 \times D^{-1.3308} + 1.2624 \times D^{-3.3308} + 0.8126 \times D^{-2.3308}\right) =$$
$$= 0.00252473 \times D^{-3.3308} + 0.00162517 \times D^{-2.3308} - 0.000926505 \times D^{-1.3308},$$

which for D = 0.877 m results in $J''_D > 0$.

CONCLUSIONS

The problems of conditioned extremes of functions of several variables, solved by means of the pertinent substitution using the technique that we will call "reduction of variables" (on some occasions it has also received the name of "substitution" or "elimination"), are reduced to others with the same variables or one less variable and without any restrictive condition, which greatly simplifies their resolution.

In this article various examples have been presented and at the end a practical case of Hydraulics that show, once again, the usefulness of the proposed procedure of "reduction of variables" in a large number of real cases that occur in practice engineering.

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