Strongly Homotopy Lie Algebras from Multisymplectic Geometry

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Contents

Chapter 1

Introduction

Multisymplectic geometry [1–4] considers generalizations of symplectic manifolds[1] called n-plectic manifolds. A differentiable manifold $\mathcal{M}$ is n-plectic if it is equipped with a closed non-degenerate $(n+1)$-form, which defines a n-plectic structure on $\mathcal{M}$.

In symplectic geometry, one can equip the space of functions on a symplectic manifold with the structure of a Poisson algebra by means of the symplectic form [5, 6]. It turns out that in multisymplectic geometry, the n-plectic structure present on a n-plectic manifold gives the structure of Lie-n algebra to a particular complex L constructed out of the complex of differential forms on the manifold [7–11]. Lie n-algebras are particular instances of strongly homotopy Lie algebras [12], or L_∞-algebras, in which the underlying complex is finite. Therefore, we see that the familiar Poisson algebra appearing in symplectic geometry extends to multisymplectic geometry in the form of an appropriate L_∞-algebra.

In symplectic geometry it is very important to consider symplectic manifolds admitting Lie-group smooth actions that preserve the symplectic structure. These smooth actions are hence called *symplectic*. Among all symplectic actions there exists a very important class which intuitively speaking is characterized by being generated by Hamiltonian vector fields and is hence called *Hamiltonian*. More precisely, the action of a Lie group G, with Lie algebra $\mathfrak{g}$, on a symplectic manifold $(\mathcal{M}, \omega)$ is said to be Hamiltonian if it admits a moment map, that is, a map [5, 6]:

$$\mu\colon \mathcal{M} \to \mathfrak{g}^*\,, \tag{1.1}$$

such that the following conditions are satisfied:

1. For each $x \in \mathfrak{g}$, let

   - $\mu^x : \mathcal{M} \to \mathbb{R}$, given by $\mu^x(p) \equiv < \mu(p), x >$, where $< \cdot, \cdot >$ is the natural pairing of $\mathfrak{g}$ and $\mathfrak{g}^*$.
   - v_x be the vector field generated by the one-parameter subgroup $\{e^{tx} \mid t \in \mathbb{R}\} \subseteq G$.

   Then:

   $$d\mu^x = -\iota_{v_x}\omega\,, \tag{1.2}$$

   that is, μ^x is a Hamiltonian function for the vector field v_x.

2. The map μ is equivariant with respect to the given action and the coadjoint action Ad^* of G on $\mathfrak{g}^*$.

[1] A symplectic manifold is adifferentiable manifold equipped with a closed non-degenerate two-form

In particular, from its definition we see that if an action is Hamiltonian then its infinitesimally generated by Hamiltonian vector fields. The notion of Hamiltonian action on a symplectic manifold can be equivalently defined[2] in terms of a comoment map, that is, a Lie algebra homomorphism:

$$\mu^* \colon \mathfrak{g} \to C^\infty(\mathcal{M}) \,, \tag{1.3}$$

such that $d\mu^*(x) = -\iota_{v_x}\omega \,, \quad x \in \mathfrak{g}, v_x \in \mathcal{M}$. That is, $\mu^*(x)$ is the Hamiltonian vector field of $v_x \in \mathfrak{X}(M)$. Associated to the Hamiltonian action of a Lie group G on a symplectic manifold $(\mathcal{M}, \omega)$ with moment map μ we can define the concept of *symplectic reduction* or *Marsden Weinstein quotient* [13]. Under some suitable assumptions, this construction gives a new smooth symplectic manifold out of $(\mathcal{M}, \omega)$ as the quotient by G of the preimage of $0 \in \mathfrak{g}^*$ by μ.

Hamiltonian actions of a Lie group G on a symplectic manifold and the associated symplectic reductions are of utmost importance in various areas of geometry and mathematical physics. For example, the concept of moment map and symplectic reduction plays an important role in the description of various moduli spaces of relevance in mathematics and mathematical physics. Particular instances include the seminal description of flat connections on a Riemann surface given in reference [14], the Donaldson−Uhlenbeck−Yau equations [15, 16] or the Einstein equations for Kähler metrics, see the book [17] for more details and further references.

Remarkably enough, the notion of Hamiltonian action can be also defined for the action of a Lie group acting on an n-plectic manifold by defining the so-called *homotopy moment map* [18], a generalization of the comoment map construction (1.3) for n-plectic manifolds. Intuitively speaking, it consists on a L_∞-morphism:

$$f \colon \mathfrak{g} \to L \,, \tag{1.4}$$

that lifts, in a suitable sense, the map from $\mathfrak{g}$ to the set of Hamiltonian vector fields which is assumed to exist from the onset. We can define then, following [18], the Hamiltonian action of a Lie group G on a n-plectic manifold as an action that preserves the n-plectic structure and in addition admits a homotopy moment map.

This master thesis is devoted to the study of L_∞-morphisms between Lie-n algebras constructed on n-plectic manifolds, as well as the study of homotopy moment maps on n-plectic manifolds equipped with the Hamiltonian action of a Lie group.

One feature of multisymplectic geometry, is that it admits a natural operation which has no counterpart in symplectic geometry, namely the wedge product: let (M_a, ω_a) be a n_a-plectic manifold, and similarly let (M_b, ω_b) be a n_b-plectic manifold. Then

$$(\tilde{M}, \tilde{\omega}) := (M_a \times M_b, \omega_a \wedge \omega_b) \tag{1.5}$$

is also a multisymplectic manifold, since ω is a non-degenerate $(n_a + n_b + 2)$-form. Notice that while this structure is natural and always well-defined, the structure on $\tilde{M}$ that is familiar from symplectic geometry – namely the sum $\omega_a + \omega_b$ – is of little use since it is not a form of well-defined degree except in the case $n_a = n_b$.

The main goal of this thesis is to show that both the L_∞-algebra of observables and homotopy moment maps are well-behaved with respect to the above wedge product operation in multisymplectic geometry.

More precisely, assuming that a Lie group G_C, with Lie algebra $\mathfrak{g}_C$, acts on (M_C, ω_C) with homotopy moment map $f^C : \mathfrak{g}_C \to L_\infty(M_C, \omega_C)$, for $C = a\,, b$:

[2] For a connected Lie group G.

1. We construct a homotopy moment map

$$F : \mathfrak{g}_a \oplus \mathfrak{g}_b \to L_\infty(\tilde{M}, \tilde{\omega})$$

   for the product manifold $\left(\tilde{M}, \tilde{\omega}\right)$, out of the homotopy moment maps f^C for the individual factors.

2. We construct an L_∞-embedding

$$H : L_\infty(M_a, \omega_a) \oplus L_\infty(M_b, \omega_b) \to L_\infty(\tilde{M}, \tilde{\omega})$$

   from the direct sum of the L_∞-algebras of the factors, to the L_∞-algebra of the product manifold.

We will see that the two questions addressed above are closely related. Indeed, rather than approaching directly question (2), we first construct F as in question (1), and using its explicit formula we are able to make an educated guess for H as in question (2) so that the following diagram of L_∞-morphisms commutes:

$$\begin{tikzcd}
L_\infty\left(M_a, \omega_a\right) \oplus L_\infty\left(M_b, \omega_b\right) \arrow[r, "\mathrm{H}"] & L_\infty\left(\tilde{M}, \tilde{\omega}\right) \\
\mathfrak{g}_a \oplus \mathfrak{g}_b \arrow[u, "f^a \oplus f^b"] \arrow[ur, "F"'] &
\end{tikzcd} \tag{1.6}$$

We explicitly construct the homotopy moment map F out of f^a and f^b (see theorem 6.4.3), making use of the machinery developed in [19, 20], and we compare our construction with the one given by [18] for homotopy moment maps arising from equivariant cocycles. In addition, making an educated guess based on the existence of F, we will explicitly construct the map:

$$L_\infty\left(M_a, \omega_a\right) \oplus L_\infty\left(M_b, \omega_b\right) \to L_\infty\left(\tilde{M}, \tilde{\omega}\right) , \tag{1.7}$$

see theorem 6.6.2. The study of differentiable manifolds equipped with closed non-degenerate forms can be justified from different points of views in mathematics as well as in physics. Standard motivations correspond to the important role that symplectic and multisymplectic manifolds play in classical mechanics, classical field theory and also in the corresponding quantization procedures. In addition, n-plectic manifolds[3] may be physically relevant on another level: the space-time manifold that describes the universe, at least up to some energy scale, could have the structure of an n-plectic manifold.

Such possibility naturally arises in Superstring Theory [21–31], a very promising candidate theory for the quantum description of all the known interactions of nature. Superstring theory implies the existence of several differential forms defined on the space-time manifold, some of them closed, corresponding to field strengths of the Ramond-Ramond and the Neveu-Schwarz Neveu-Schwarz forms of the corresponding Supergravity. Therefore the space-time manifold in Superstring theory is going to be at least a pre-n-plectic manifold. The non-degeneracy properties of the forms will depend on the particular solution to be considered.

[3]Maybe dropping the non-degeneracy condition on the $(n+1)$-form. The corresponding manifold is then called pre-n-plectic. Note however that most of the results about n-plectic manifolds can be extended in a suitable way to pre-n-plectic manifolds.

The space-time that we observe is four-dimensional, yet Superstring theory predicts that the space-time must be ten-dimensional. In order to fix this apparent contradiction, several mechanisms have been proposed in the literature [32–37]. One of them, the Kaluza-Klein reduction [34], consists in assuming that the space-time manifold $\mathcal{M}$ is locally the product of a four-dimensional non-compact manifold $\mathcal{M}_4$ and a six-dimensional compact manifold $\mathcal{M}_6$

$$\mathcal{M} = \mathcal{M}_4 \times \mathcal{M}_6 \,, \tag{1.8}$$

small enough to not be accessible in current high-energy experiments. Superstring theory constrains the different manifolds $\mathcal{M}_6$ that we can consider as compact manifolds [37]. In particular, for supersymmetric compactifications, the existence of one or several globally defined spinors on the compact manifold implies the existence of globally defined forms, which, depending on the details of the compactification, may be closed and non-degenerate. As an example, we can consider M-theory [38], closely related to Superstring Theory, which is a theory that predicts the space-time manifold to be eleven-dimensional. The fluxless compactification of such theory on a seven dimensional compact manifold $\mathcal{M}_7$ implies that $\mathcal{M}_7$ must be a manifold of G_2-holonomy [34, 39–41]. Therefore, it has a globally defined, closed and non-degenerate three-form [42] and thus it is a two-plectic manifold.

It is worth pointing out that the interpretation of the Lie-n algebras associated to the space-time manifold or the compactification manifold is not known, and it would be interesting to find out if it encodes any physical information about the theory itself. Notice that L_∞-algebras have appeared already in Superstring Theory and Supergravity. For example, the algebra of states in the Fock space of closed String Field Theory is a strongly homotopy Lie algebra [43]. For more applications of L_∞-algebras to Superstring Theory and Supergravity the interested reader may consult [44–52]. It is clear then that multisymplectic geometry and L_∞-algebras play an important role in theoretical physics and in particular in Superstring Theory and Supergravity, and thus more effort is needed in order to uncover the role that these mathematical structures play in the theories that describe the fundamental interactions of nature.

The outline of this work is as follows. In **chapter 2** we introduce some background material relevant for the rest of the paper, which includes basics of fiber bundles, Lie groups and Courant algebroids. It is intended for non-experts, perhaps interested physicists, and therefore can be skipped by experts. In **chapter 3** we consider symplectic manifolds, moment maps and the Marsden-Weinstein quotient. In **chapter 4** we introduce some background on algebras, coalgebras and categories. In **chapter 5** we introduce L_∞-algebras and define L_∞-morphisms in an independent way, not related yet to multisymplectic geometry, giving explicit formulae relating $L_\infty[1]$-algebras and L_∞-algebras. **Chapter 6** contains the new results present in this thesis. We first introduce n-plectic manifolds and connect them to L_∞-algebras. Then we introduce, closely following [18], the concept of homotopy moment map. In section 6.2 we obtain specific conditions under which two n-plectic manifolds with strictly isomorphic Lie-n algebras are symplectomorphic. In section 6.3, we study the construction of an homotopy moment map for a product manifold assuming that the factors are n-plectic manifolds equipped with the corresponding homotopy moment maps. Then in section 6.5 we specialize to the case of iterated powers of the same multisymplectic form, i.e. (M, ω^m), displaying explicit formulae for the case (M, ω^2) and discussing Hyperkähler manifolds as an example. In section 6.6 we construct the L_∞-embedding H (by L_∞-embedding we mean an L_∞-morphism whose first component H_1 is injective). We do this in theorem 6.6.2, using the formulae for F as a guide.

Chapter 2

Background material

In this section we introduce some basic material which will be used through the rest of the document. Standard references for this chapter are [53–56].

2.1 Manifolds and Lie groups

A topological space $\mathcal{M}$ is said to be Hausdorff or T_2 if for every pair points $p, q \in \mathcal{M}$ there exist neighbourhoods $\mathcal{U}(p), \mathcal{U}(q)$ of p and q, such that $\mathcal{U}(p) \cap \mathcal{U}(q) = \{\emptyset\}$. In addition, $\mathcal{M}$ is a second-countable space if it has a countable base, that is, if there exists a countable collection $\{\mathcal{U}_i\}_{i=1}^{\infty}$ of open sets such that any open set in $\mathcal{M}$ can be written as a union of open sets in the collection $\{\mathcal{U}_i\}_{i=1}^{\infty}$. A *coordinate chart* on $\mathcal{M}$ is a pair $(\mathcal{U}, \phi)$, where $\mathcal{U}$ is an open subset of $\mathcal{M}$ and ϕ is a homeomorphism of $\mathcal{U}$ onto an open subset of $\mathbb{R}^n$.

Definition 2.1.1. Let $\mathcal{M}$ be a Hausdorff, second-countable, topological space. A *differentiable structure* on $\mathcal{M}$ is a collection of coordinate charts $\{\mathcal{U}_\alpha, \phi_\alpha\}_{\alpha \in I}$ on $\mathcal{M}$ such that the following conditions hold

1. $\mathcal{M} = \bigcup_{\alpha \in I} \mathcal{U}_\alpha$
2. $\phi_\alpha(\mathcal{U}_\alpha)$ is an open set of $\mathbb{R}^n$ for all $\alpha \in I$ and for each pair $\alpha, \beta \in I$, $\phi_\beta \circ \phi_\alpha^{-1}$ is a differentiable[1] mapping of $\phi_\alpha(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$ onto $\phi_\beta(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$.
3. The collection $(\mathcal{U}_\alpha, \phi_\alpha)_{\alpha \in I}$ is a maximal family of open charts for which conditions 1 and 2 hold. The family $(\mathcal{U}_\alpha, \phi_\alpha)_{\alpha \in I}$ is then called the *maximal atlas* of $\mathcal{M}$.

Definition 2.1.2. A *differentiable manifold* of dimension n is a Hausdorff, second-countable, topological space equipped with a differentiable structure of dimension n.

If $\mathcal{M}$ is a manifold, a *local chart* or *local coordinate system* on $\mathcal{M}$ is a pair $(\mathcal{U}_\alpha, \phi_\alpha)$ where $\alpha \in I$. For every $p \in \mathcal{U}_\alpha, \alpha \in I$, $\mathcal{U}_\alpha$ is called a *coordinate neighbourhood* of p and the numbers $\phi_\alpha(p) = (\mathrm{x}^1(p), \ldots, \mathrm{x}^n(p))$ are the local coordinates of p. Condition 3 is not essential in the definition of a manifold, since if only 1 and 2 are satisfied, the family $(\mathcal{U}_\alpha, \phi_\alpha)_{\alpha \in I}$ can be extended in a unique way to a family of charts such that 1, 2 and 3 are fulfilled.

Since a manifold $\mathcal{M}$ is locally homeomorphic to $\mathbb{R}^n$, they share the same local topological properties. In particular, manifolds are locally compact and locally connected. That means, respectively, that every point $p \in \mathcal{M}$ has a compact neighbourhood and a connected neighbourhood. Using that the topology of $\mathcal{M}$ has a countable basis and it is locally compact, it can be shown that $\mathcal{M}$ is paracompact, that is,

[1] By differentiable we will always mean, unless otherwise stated, infinitely differentiable or C^∞.

every open cover of $\mathcal{M}$ admits a locally finite refinement. Paracompactness is a sufficient condition for partitions of unit to exist, and therefore $\mathcal{M}$ admits a Riemannian metric, which implies in particular that it is metrizable. Schematically we can write:

$$\mathcal{M}\colon \text{Hausdorff} \oplus \mathrm{T}_2 \oplus \text{locally}\mathbb{R}^n \to \text{Paracompact and metrizable}\,. \tag{2.1}$$

In order to give some intuition or justification to the various conditions included in the definition of a manifold, let us take an example from Physics, in particular from General Relativity. In the context of General Relativity, the space-time is usually described as an n-dimensional differentiable manifold $\mathcal{M}$. In that context, the Hausdorff condition is natural since it is experimentally observed. On the other hand, the gravitational interaction is described by a Lorentzian metric g on $\mathcal{M}$. The condition that $\mathcal{M}$ is paracompact ensures the existence of a partition of unity, which in turn ensures the existence of a Riemannian metric on $\mathcal{M}$. When $\mathcal{M}$ is non-compact, it also ensures the existence of a Lorentzian metric[2], as required in General Relativity. The second-countable condition is a reasonable assumption for topological spaces locally homeomorphic to $\mathbb{R}^n$, since otherwise the space would not be adapted to be locally *like* $\mathbb{R}^n$. If we want to use $\mathcal{M}$ to describe the space-time, this is a very natural assumption since it is experimentally observed that, at least at some scales, the space-time actually locally *looks like* $\mathbb{R}^n$.

A function $f : \mathcal{M} \to \mathbb{R}$ is differentiable at $p \in \mathcal{U}_\alpha \subset \mathcal{M}$ if $f \circ \phi_\alpha^{-1} : \phi_\alpha(\mathcal{U}_\alpha) \subset \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\phi_\alpha(p) \in \mathbb{R}^n$. A function f is called differentiable if it is differentiable at every point $p \in \mathcal{M}$. We denote by $C^\infty(\mathcal{M})$ the set of differentiable functions from $\mathcal{M}$ into $\mathbb{R}$ and by $C^\infty(\mathcal{M}, p)$ the set of functions from $\mathcal{M}$ into $\mathbb{R}$ differentiable at $p \in \mathcal{M}$.

Let $\mathcal{M}$ be a manifold with differentiable structure $\{\mathcal{U}_\alpha, \phi_\alpha\}_{\alpha\in I}$. There are three basic and equivalent ways to define the tangent space $T_p\mathcal{M}$ of a differentiable manifold $\mathcal{M}$ at a point $p \in \mathcal{M}$:

1. Let $\mathfrak{T}_p$ be the set of all pairs (ϕ_α, u), where $p \in \mathcal{U}_\alpha$ and $u \in \mathbb{R}^n$. We define an equivalence relation $\sim$ on $\mathfrak{T}_p$ by declaring $(\phi_\alpha, u) \sim (\phi_\beta, v)$ if and only if :
$$d_{\phi_\alpha(p)}\left(\phi_\beta \circ \phi_\alpha^{-1}\right)(u) = v\,, \tag{2.2}$$
for every other coordinate chart $(\mathcal{U}_\beta, \phi_\beta)$ such that $p \in \mathcal{U}_\beta$. The equivalence class of (ϕ_α, u) will be denoted by $[\phi_\alpha, u]$. The set $T_p\mathcal{M} \equiv \mathfrak{T}_p/\sim$ is then the tangent space at the point $p \in \mathcal{M}$. If $\{e_i\}, i = 1, \dots, n$, is the canonical basis of $\mathbb{R}^n$ we define the partial derivatives respect to x^i by:
$$\frac{\partial}{\partial \mathrm{x}^i} = [\phi_\alpha, e_i]\,, \qquad i = 1, \dots, n\,. \tag{2.3}$$

2. A curve in $\mathcal{M}$ is a map $c : [0,1] \to \mathcal{M}$. A curve c is differentiable at $t_0 \in (0,1), c(t_0) \in \mathcal{U}_\alpha \subset \mathcal{M}$, if $\phi_\alpha \circ c : [0,1] \to \mathbb{R}$ is differentiable at t_0. Let be $\mathfrak{T}_p$ the set of all the curves in $\mathcal{M}$ passing through $p \in \mathcal{U}_\alpha$. We define the following equivalence relation $\sim$: two curves $c_1(t)$ and $c_2(t)$ on $\mathcal{M}$ passing through p are related by $\sim$ if and only if:
$$\partial_t\left(\phi_\alpha \circ c_1\right)(t_0) = \partial_t\left(\phi_\alpha \circ c_2\right)(t_0)\,, \tag{2.4}$$
where $t_0 \in (0,1)$ is a fixed real number. We denote by $[c]$ the class of equivalence of c. Then the tangent space is $T_p\mathcal{M} \equiv \mathfrak{T}_p/\sim$.

3. A derivation at $p \in \mathcal{M}$ is a linear application $D : C^\infty(\mathcal{M}) \to \mathbb{R}$ such that
$$D(fg)|_p = f(p)\,D(g)|_p + g(p)D(f)|_p\,. \tag{2.5}$$

[2] When $\mathcal{M}$ is compact, the topological obstruction for the existence of a Lorentzian metric is given by the vanishing of its Euler characteristic.

for every $f, g \in C^\infty(\mathcal{M}, p)$. We denote the space of derivations at $p \in \mathcal{M}$ by $\mathrm{Der}(\mathcal{M}, p)$. Then we have $T_p\mathcal{M} \equiv \mathrm{Der}(\mathcal{M}, p)$.

Let $\mathcal{M}$ be a differentiable manifold of dimension n with atlas $(\mathcal{U}_\alpha, \phi_\alpha)_{\alpha\in I}$ and let $\mathcal{N}$ be a differentiable manifold of dimension m with atlas $(\mathcal{V}_\beta, \psi_\beta)_{\beta\in J}$. Let Φ be a mapping from $\mathcal{M}$ to $\mathcal{N}$. The map Φ it is said to be differentiable at a point $p \in \mathcal{U}_\alpha$ if the map $\psi_\beta \circ \Phi \circ \phi_\alpha^{-1} : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $\phi_\alpha(p) \in \mathbb{R}^n$. The map Φ is said to be differentiable if it is differentiable at every point p in $\mathcal{M}$. The map Φ is said to be a diffeomorphism if it is a differentiable biyective map with differentiable inverse. The map Φ is said to be a local diffeomorphism if for every $p \in \mathcal{M}$ there exists an open set $\mathcal{U} \in \mathcal{M}$ containing p such that Φ restricted to $\mathcal{U}$ is a diffeomorphism into its image, which is automatically open in $\mathcal{N}$.

Let us define $\Phi(p) := q \in \mathcal{V}_\beta$. The differential $d_p\Phi$ of Φ at a point $p \in \mathcal{M}$ is a linear map $d_p\Phi : T_p\mathcal{M} \to T_q\mathcal{N}$ which can be naturally defined, for each of the equivalent definitions of the tangent space introduced above, as follows:

1. $d_p\Phi : [\phi_\alpha, u] \mapsto \left[\psi_\alpha, d_{\phi_\alpha(p)}\left(\psi_\alpha \circ \Phi \circ \phi_\alpha^{-1}\right)(u)\right]$
2. $d_p\Phi : [c] \mapsto [\Phi \circ c]$
3. $d_p\Phi : D \mapsto \Phi_* D$, where $\Phi_* D(h) = D(h \circ \Phi)$ for every $h \in C^\infty(\mathcal{N}, q)$.

We are ready to introduce the tangent bundle of $\mathcal{M}$, which is a special instance of vector bundle, which will be defined in section 2.2. Let us consider the set

$$T\mathcal{M} = \{(p, v) : p \in \mathcal{M}\,, v \in T_p\mathcal{M}\} = \bigcup_{p\in\mathcal{M}} \{p\} \times T_p\mathcal{M}\,. \tag{2.6}$$

There is a natural projection map $\pi : T\mathcal{M} \to \mathcal{M}$ given by $\pi(p, v) = p$ for every $(p, v) \in T\mathcal{M}$. The tangent bundle $T\mathcal{M}$ admits a natural topology and differentiable structure for which π is a continuous and in fact smooth map. We topologize $T\mathcal{M}$ by taking as open sets the sets of the form $\pi^{-1}(\mathcal{U}_\alpha)$, where $\mathcal{U}_\alpha \subset \mathcal{M}$ is an open set of the maximal atlas of $\mathcal{M}$. We define a differentiable structure on $T\mathcal{M}$ by defining on every open set $\pi^{-1}(\mathcal{U}_\alpha)$ the following coordinates $\tilde{\phi}_\alpha$:

$$\tilde{\phi}_\alpha(p, v) = (\phi_\alpha(p), d_p\phi_\alpha(v)) \in \mathbb{R}^{2n}\,, \tag{2.7}$$

where $p \in \mathcal{U}_\alpha \subset \mathcal{M}$ and $v \in T_p\mathcal{M}$. Therefore, $T\mathcal{M}$ is a $2n$-dimensional manifold which is in particular a vector bundle of rank n and fibre at a point $p \in \mathcal{M}$ given by the vector space $T_p\mathcal{M}$.

Smooth sections[3] of $T\mathcal{M}$ are smooth maps $v\colon \mathcal{M} \to T\mathcal{M}$ such that $\pi \circ v(p) = p$ for all $p \in \mathcal{M}$. A smooth **vector field** on $\mathcal{M}$ is a smooth section of $T\mathcal{M}$, and the $C^\infty(\mathcal{M})$-module of all vector fields in $\mathcal{M}$ is denoted by $\mathfrak{X}(M)$. Vector fields on a manifold $\mathcal{M}$ can be integrated along curves on $\mathcal{M}$. Let $v \in \mathfrak{X}(M)$ be a vector field on $\mathcal{M}$ and fix a point $p \in \mathcal{M}$. Then, there exists a positive number ϵ and a unique curve $\gamma\colon [-\epsilon, \epsilon] \to \mathcal{M}$ with parameter t such that:

$$\partial_t\gamma|_0 = v|_p\,, \tag{2.8}$$

and:

$$\partial_t\gamma(t) = v|_{\gamma(t)}\,, \tag{2.9}$$

for every $t \in (-\epsilon, \epsilon)$. A vector field $v \in \mathfrak{X}(\mathcal{M})$ is said to be complete if the parameter of each integral curve extends to $(-\infty, \infty)$. A complete vector field $v \in \mathfrak{X}(M)$ generates a one-parameter family of diffeomorphisms $\rho_t : \mathcal{M} \to \mathcal{M}\,, t \in \mathbb{R}$, as follows. For each $p \in \mathcal{M}$ and $t \in \mathbb{R}$, we define $\rho_t(p)$ to be the value in $\mathcal{M}$ at t of the unique integral curve of v passing through p at $t = 0$.

[3] See also definition (2.2.3).

Definition 2.1.3. Let $v \in \mathfrak{X}(\mathcal{M})$ be a complete vector field on $\mathcal{M}$. The *one-parameter group of diffeomorphisms* $\{\rho_t\}_{t\in\mathbb{R}}$ associated to $v \in \mathfrak{X}(M)$ is defined as:

$$\begin{aligned} \rho_t\colon \mathcal{M} &\to \mathcal{M}\,, \\ p &\mapsto \gamma_{v_p}(t)\,, \end{aligned} \tag{2.10}$$

where $\gamma_{v_p}\colon \mathbb{R} \to \mathcal{M}$ is the unique smooth complete curve in $\mathcal{M}$ such that $\partial_t\gamma|_0 = v|_p$ and $\partial_t\gamma = v|_{\gamma(t)}$. Equivalently, we define the **flow** of the complete vector field $v \in \mathfrak{X}(\mathcal{M})$ as:

$$\begin{aligned} \varphi_v\colon \mathbb{R}\times\mathcal{M} &\to \mathcal{M}\,, \\ (t,p) &\mapsto \rho_t(p)\,, \end{aligned} \tag{2.11}$$

Remark 2.1.4. It can be can be seen that the map $\mathbb{R} \to \mathrm{Diff}\,(\mathcal{M})$ defined by $t \mapsto \rho_t$ is a group homomorphism. Therefore, every complete vector field on $\mathcal{M}$ defines a smooth action of $\mathbb{R}$ on $\mathcal{M}$ (see section 2.1.2 for more details).

At every point $p \in \mathcal{M}$ we denote the dual space of $T_p\mathcal{M}$ as $T^*_p\mathcal{M}$. Elements of $T^*_p\mathcal{M}$ are called one-forms at the point p. Similarly, the dual bundle of $T\mathcal{M}$ is denoted by $T^*\mathcal{M}$. Sections of $T^*\mathcal{M}$ one-form fields on $\mathcal{M}$. They correspond simply to a smooth choice of one-form in $T^*_p\mathcal{M}$ at every point $p \in \mathcal{M}$. The set of all the one-form fields in $\mathcal{M}$ is denoted by $\Omega^1(\mathcal{M})$ or equivalently by $\Gamma(T^*\mathcal{M})$. Analogously, an element $\mathfrak{T}_p$ of $(T_p\mathcal{M})^{\otimes s} \otimes \left(T^*_p\mathcal{M}\right)^{\otimes r}$ is a (r, s) tensor[4] and a section $\mathfrak{T}$ of $\Gamma\left((T\mathcal{M})^{\otimes s} \otimes (T^*\mathcal{M})^{\otimes r}\right)$ is a (r, s) tensor field on $\mathcal{M}$.

Of utmost importance in differential geometry are the tensor algebra[5] $(\mathrm{T}\,(\mathcal{M})\,,\otimes)$ and the exterior algebra of differential forms $(\Lambda\,(\mathcal{M})\,,\wedge)$. Let $\mathrm{T}_{(r,s)}\,(\mathcal{M})$ denote the set of all tensor fields on $\mathcal{M}$ of type (r, s), and let $\Omega^k\,(\mathcal{M})$ denote the set of all k-form fields on $\mathcal{M}$. Then we have[6]

$$\mathrm{T}\,(\mathcal{M}) = \sum_{r,s=1}^{\infty} \mathrm{T}_{(r,s)}\,(\mathcal{M})\;, \qquad \Lambda\,(\mathcal{M}) = \sum_{k=1}^{\infty} \Omega^k\,(\mathcal{M})\;. \tag{2.12}$$

Note that the infinite sum in the definition of $\Lambda\,(\mathcal{M})$ is only formal; for finite-dimensional manifolds it will contain only a finite number of terms. With this definition, $\Lambda(M)$ is a $\mathbb{Z}$-graded commutative algebra.

There are several important operators that can be defined on $\mathrm{T}\,(\mathcal{M})$ and $\Lambda\,(\mathcal{M})$. Here we will consider the interior product ι_v, the exterior derivative or de Rham differential d and the Lie derivative $\mathcal{L}_v$, $v \in \Gamma\,(T\mathcal{M})$.

The interior product ι_v The interior product $\iota_v\colon \Omega^i\,(\mathcal{M}) \to \Omega^{(i-1)}\,(\mathcal{M})$ is a -1 degree derivation on the exterior algebra of differential forms $\Lambda\,(\mathcal{M})$. It is defined to be the contraction of a differential form with a vector field $v \in \mathcal{M}$ as follows:

$$(\iota_v\omega)\left(v_1,\ldots,v_{(i-1)}\right) = \omega\left(v,v_1,\ldots,v_{(p-1)}\right)\,, \qquad \forall\; v_1,\ldots,v_{(p-1)} \in \mathfrak{X}\,(\mathcal{M})\;. \tag{2.13}$$

The interior product is the unique derivation of degree minus one on the exterior algebra such that on one-forms corresponds to the natural pairing of one-forms and vectors.

[4]For the definition of tensor product see 4.1.9.

[5]For the definition of algebra see 4.1.1.

[6]For more details and general definitions, see section 4.1.

The exterior derivative d The exterior derivative d is defined to be the unique $\mathbb{R}$-linear mapping $d\colon \Omega^i(\mathcal{M}) \to \Omega^{(i+1)}(\mathcal{M})$ such that

- df is the differential of f for every function $f \in C^\infty(\mathcal{M})$.
- $d \circ df = 0$ for every function $f \in C^\infty(\mathcal{M})$.
- $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (1)^p \alpha \wedge d\beta$, where α is a p-form and β is any form.

Since the second defining property holds in more generality, that is, $d \circ d\alpha = 0$ for any p-form α, it is usually written as $d^2 = d \circ d = 0$.

The Lie derivative $\mathcal{L}_v$ The Lie derivative can be defined acting on tensor fields of any type (r, s), that is, it has a well defined action on $\mathrm{T}(\mathcal{M})$. Intuitively, the Lie derivative $\mathcal{L}_v$ evaluates the change of a tensor field along the flow of the vector field v. It is defined point-wise as follows

$$(\mathcal{L}_v \mathfrak{T})_p = \left.\frac{d}{dt}\right|_{t=0} \left(\varphi(-t,p)_* \mathfrak{T}_{\varphi_t(p)}\right) = \left.\frac{d}{dt}\right|_{t=0} (\varphi(t,p)^* \mathfrak{T})_p \,, \tag{2.14}$$

where $\mathfrak{T}$ is a (r, s) tensor field on $\mathcal{M}$ and $p \in \mathcal{M}$. It can be checked that with the definition (2.14) $\mathcal{L}_v \mathfrak{T}$ is again a (r, s) tensor field on $\mathcal{M}$. We now give an algebraic definition. The algebraic definition for the Lie derivative of a tensor field follows from the following four axioms

- $\mathcal{L}_v f = v(f)$ for all $f \in C^\infty(\mathcal{M})$.
- The Lie derivative $\mathcal{L}_v$ obeys the Leibniz rule. That is, for any tensor fields $\mathfrak{S}$ and $\mathfrak{T}$, we have

$$\mathcal{L}_v (\mathfrak{S} \otimes \mathfrak{T}) = (\mathcal{L}_v \mathfrak{S}) \otimes \mathfrak{T} + \mathfrak{S} \otimes (\mathcal{L}_v \mathfrak{T}) \,, . \tag{2.15}$$

- The Lie derivative, when applied to forms, obeys the Leibniz rule with respect to contraction

$$\mathcal{L}_v(\mathfrak{T}(Y_1, \ldots, Y_n)) = (\mathcal{L}_v \mathfrak{T})(Y_1, \ldots, Y_n) + \mathfrak{T}((\mathcal{L}_v Y_1), \ldots, Y_n) + \cdots + T(Y_1, \ldots, (\mathcal{L}_v Y_n)) \tag{2.16}$$

- The Lie derivative, when applied to forms, commutes with the de Rham differential d, that is

$$[\mathcal{L}_v, d] = 0 \,, \tag{2.17}$$

The Lie derivative $\mathcal{L}_v$ can be compactly written as

$$\mathcal{L}_v = \iota_v \circ d + d \circ \iota_v \,, \tag{2.18}$$

which is known as the *Cartan formula.*

2.1.1 Cartan calculus

Let us denote by $\mathfrak{X}(\mathcal{M})$ the $C^\infty(\mathcal{M})$-module of vector fields on $\mathcal{M}$. Then

$$\mathfrak{X}^\bullet(\mathcal{M}) = \bigoplus_{k=0}^{\dim \mathcal{M}} \Lambda^k \mathfrak{X}(\mathcal{M}) \,, \tag{2.19}$$

is a graded commutative algebra, the so-called graded commutative algebra of multivector fields, where the corresponding algebra product is given by the wedge product, denoted by $\wedge$. $\mathfrak{X}^\bullet(\mathcal{M})$ can be equipped with a degree minus one Lie bracket $[\cdot,\cdot] : \mathfrak{X}^\bullet(\mathcal{M}) \times \mathfrak{X}^\bullet(\mathcal{M}) \to \mathfrak{X}^\bullet(\mathcal{M})$ that satisfies the (graded) Leibniz rule with respect to the algebra product, that is, the wedge product. $[\cdot,\cdot]$ is given by

$$[u_1 \wedge \cdots \wedge u_m, v_1 \wedge \cdots \wedge v_n] = \sum_{i=1}^{m} \sum_{j=1}^{n} (-1)^{i+j} [u_i, v_j] \wedge u_1 \wedge \cdots \wedge \hat{u}_i \wedge \cdots \wedge u_m \wedge v_1 \wedge \cdots \wedge \hat{v}_j \wedge \cdots \wedge v_n ,$$

where $u_1 \wedge \cdots \wedge u_m$, $v_1 \wedge \cdots \wedge v_n \in \mathfrak{X}^\bullet(\mathcal{M})$ and $[u_i, v_j]$ is the standard Lie bracket of vector fields. This is the so-called *Schouten bracket*, and it makes $(\mathfrak{X}^\bullet(\mathcal{M}), \wedge, [\cdot,\cdot])$ into a particular instance of Gerstenhaber algebra[7].

We can define also the interior product of any decomposable multivector field, say $v_1 \wedge \cdots \wedge v_n$, with any $\beta \in \Omega^\bullet(\mathcal{M})$ is given by

$$\iota(v_1 \wedge \cdots \wedge v_n)\beta = \iota_{v_n} \cdots \iota_{v_1}\beta, \tag{2.20}$$

where $\iota_{v_i}\beta$ is the stands for the usual interior product of vector fields and differential forms. The formula for the interior product of any multivector can be obtained by extending using $C^\infty(\mathcal{M})$ linearity.

The Lie derivative $\mathcal{L}_v$ of any differential form β along any given multivector field $v \in \mathfrak{X}^\bullet(\mathcal{M})$ can be written in terms of the graded commutator of d and ι_v as follows

$$\mathcal{L}_v\beta = d\iota_v\beta - (-1)^{|v|}\iota_v d\beta , \tag{2.21}$$

where ι_v must be understood as a degree $-|v|$ operator. We will need one more identity. Let $u, v \in \mathfrak{X}^\bullet(\mathcal{M})$. Then it can be proven that

$$\iota_{[u,v]}\beta = (-1)^{(|u|-1)|v|}\mathcal{L}_u\iota_v\beta - \iota_v\mathcal{L}_u\beta . \tag{2.22}$$

The graded commutative algebra of multivector fields $\mathfrak{X}^\bullet(\mathcal{M})$ together with the Schouten bracket is therefore a particular instance of a Gerstenhaber algebra that can be constructed in any differential manifold $\mathcal{M}$.

2.1.2 Lie groups

Definition 2.1.5. A *Lie group* is a smooth manifold G which is also an abstract group such that the multiplication map $\cdot$ and the inverse map are C^∞-maps.

Definition 2.1.6. A *Lie subgroup* H of a Lie group G is an abstract subgroup $H \subset G$ such that the canonical inclusion is an immersion.

Remark 2.1.7. The canonical injection is an embedding if and only if H is closed in G.

Remark 2.1.8. By Cartan's theorem, every closed subgroup of a Lie group is a Lie subgroup.

Definition 2.1.9. For a Lie group G and an element $g \in G$, *left-translation* $L_g : G \to G$ and *right-translation* $R_g : G \to G$ are smooth maps defined by:

$$L_g(h) = g \cdot h , \qquad R_g(h) = h \cdot g , \qquad \forall h \in G . \tag{2.23}$$

The maps:

[7] See definition 4.1.13.

$$\mathfrak{L}\colon g \to L_g\,, \qquad \mathfrak{R}\colon g \to R_g\,, \tag{2.24}$$

are homomorphisms from G into the diffeomorphism group Diff(G) of G. In other words, we have:

$$(L_g)^{-1} = L_{g^{-1}}\,, \qquad L_{g_1} \circ L_{g_2} = L_{g_1 g_2}\,, \tag{2.25}$$

and similarly for R_g. Notice also that left and right translations commute.

Example 2.1.10. The group $\mathrm{Gl}(n,\mathbb{R})$ of $n\times n$ invertible real matrices is a real Lie group of dimensions n^2. Its differentiable structure its induced from the canonical one of $\mathbb{R}^{n^2}$ after identifying $\mathrm{Mat}(n,\mathbb{R}) \simeq \mathbb{R}^{n^2}$ and after noticing that by definition $\mathrm{Gl}(n,\mathbb{R})$ is an open subset of $\mathbb{R}^{n^2}$ of dimension n^2.

Definition 2.1.11. A vector field $X \in \mathfrak{X}(G)$ is called **left-invariant** if $L_{g*}X = X$ for all $g \in G$. Likewise, a vector field $X \in \mathfrak{X}(G)$ is called **right-invariant** if $R_{g*}X = X$ for all $g \in G$.

We will denote the set of left-invariant vector fields by $\mathfrak{X}^{\mathrm{L}}(G)$[8]. Standard addition of vector fields and scalar multiplication by real numbers make $\mathfrak{X}^{\mathrm{L}}(G)$ into a real vector space.

Lemma 2.1.12. *The vector space of left-invariant vector fields $\mathfrak{X}^{\mathrm{L}}(G)$ is closed under the Lie bracket operation given by the standard commutator of vector fields $[\cdot,\cdot] : \mathfrak{X}(G)\times\mathfrak{X}(G) \to \mathfrak{X}(G)$ and hence $(\mathfrak{X}^L(G), [\cdot,\cdot])$ is a Lie sub-algebra of $(\mathfrak{X}(G), [\cdot,\cdot])$.*

Proof. Follows from the identity:

$$L_{g*}[v_1, v_2] = [L_{g*}v_1, L_{g*}v_2]\,, \qquad \forall\, v_1, v_2 \in \mathfrak{X}(\mathcal{M})\,. \tag{2.26}$$

□

The vector space of left-invariant vector fields $\mathfrak{X}^{\mathrm{L}}(G)$ equipped with $[\cdot,\cdot]$ is defined to be the Lie algebra $\mathfrak{g} := (\mathfrak{X}^{\mathrm{L}}(G), [\cdot,\cdot])$ of G. Left-invariant vector fields are completely determined by their value at one point, say $g_0 \in G$, since they can always be unambiguously reconstructed by left-translation, i.e.:

$$v^L|_g = (L_{g g_0^{-1}})_* v^L_{g_0}\,, \qquad \forall\, v^L \in \mathfrak{X}^L(\mathcal{M})\,, \tag{2.27}$$

where $g \in G$. Since Lie groups have a distinguished point, namely the identity element $e \in G$, left-invariant vector fields $\mathfrak{X}^{\mathrm{L}}(G)$ can be canonically identified with elements of the vector space T_eG. The tangent space at the identity T_eG equipped with the binary operation induced by the standard bracket $[\cdot,\cdot]$ of left-invariant vector fields is isomorphic to the Lie algebra $\mathfrak{g}$ of G. This in turn proves that the dimension of the Lie algebra $\mathfrak{g}$ is equal to the dimension of G. Similar remarks apply to set of right-invariant vector fields $\mathfrak{X}^{\mathrm{R}}(G)$. Under some hypothesis, Lie algebras determine completely the corresponding Lie group.

Theorem 2.1.13. *Every finite-dimensional real Lie algebra $\mathfrak{g}$ is isomorphic to the Liea algebra of some real Lie group G. If G is connected and simply connected then $\mathfrak{g}$ determines G up to isomorphism of Lie groups.*

This is the so-called *Lie's third theorem*, and its proof can be found for example in reference [57]. For general Lie groups we have the following theorem.

Theorem 2.1.14. *Two Lie groups are locally isomorphic if and only if their Lie algebras are isomorphic.*

Proposition 2.1.15. *Any Lie group G is parallelizable, that is $TG \simeq G \times \mathfrak{g}$.*

[8]In the following discussion, similar remarks apply to the set of right-invariant vector fields. It is standard in the literature to consider left-invariant vector fields.

Proof. A basis e_i, $i = 1, \dots, n$, of $\mathfrak{g}$ and extending it over G by left-translation, we obtain n everywhere non-zero and independent vector fields on G. $\square$

Definition 2.1.16. A **one-parameter subgroup** of a Lie group G is an injective smooth homomorphism $\phi\colon (\mathbb{R}, +) \to G$.

Hence, $\phi\colon \mathbb{R} \to G$ is a smooth curve that satisfies $\phi(s+t) = \phi(s)\phi(t)$ and $\phi(0) = e$. For example, e^{it} is a one-parameter subgroup of the circle $S^1 = U(1)$.

Definition 2.1.17. We make the following definitions:

- Let $x \in \mathfrak{g}$. We define $v_x \in \mathfrak{X}^L(G)$ as the unique left-invariant vector field on G such that:

$$v_x|_e = x\,. \tag{2.28}$$

- A one-parameter subgroup generated by an element $x \in \mathfrak{g}$ is the homomorphism $\Phi_x\colon \mathbb{R} \to G$ defined as:

$$\begin{aligned} \Phi_x\colon \mathbb{R} &\to G\,, \\ t &\mapsto \varphi_{v_x}(t, e)\,, \end{aligned} \tag{2.29}$$

  where $\varphi_{v_x}(t,e)$ denotes the flow associated to $v_x \in \mathfrak{X}^L(G)$. For simplicity we will sometimes denote φ_{v_x} by φ_x.

- The exponential map Exp of G is the map:

$$\begin{aligned} \mathrm{Exp}\colon \mathfrak{g} &\to G\,, \\ x &\mapsto \varphi_{v_x}(1, e)\,. \end{aligned} \tag{2.30}$$

  We will sometimes denote $\mathrm{Exp}(x)$ by e^x.

Proposition 2.1.18. *Let $g \in G$. The following equalities hold:*

$$\varphi_x(t,g) = g\mathrm{Exp}(tx)\,, \qquad \mathrm{Exp}((t+s)x) = \mathrm{Exp}(tx)\mathrm{Exp}(sx)\,, \tag{2.31}$$

where $t, s \in \mathbb{R}$.

Proposition 2.1.19. *The map $\phi \mapsto (d\phi)|_0(1)$ defines a one-to-one correspondence between one-parameter subgroups of G and T_eG.*

Proposition 2.1.20. *Let G be a compact and connected Lie group. Then the exponential map* $\mathrm{Exp}\colon \mathfrak{g} \to G$ *is surjective.*

We define now the adjoint representation of a Lie group. Every Lie group acts on itself by conjugation. Given an element $g \in G$, we define the conjugation map C_g as follows:

$$\begin{aligned} C_g : G &\to G \\ h &\mapsto ghg^{-1}\,. \end{aligned} \tag{2.32}$$

The tangent map of $C_g : G \to G$ is an un-based morphism of the tangent bundle TG whose evaluation at a point $h \in G$ is a linear map of the form:

$$(dC_g)\,|_h : T_h G \to T_{ghg^{-1}} G\,. \tag{2.33}$$

Therefore, for each $g \in G$, $(dC_g)\,|_e : T_e G \to T_e G$ is an endomorphism of the tangent space of G at the identity which can be seen to be an automorphism. In addition, $(dC_g)\,|_e$ is compatible with the Lie bracket in $T_e G \simeq \mathfrak{g}$ and therefore is a Lie-algebra automorphism of $\mathfrak{g}$.

Definition 2.1.21. We define the adjoint representation $\mathrm{Ad}\colon G \to \mathrm{Gl}(\mathfrak{g})$ of G as:

$$\begin{aligned} \mathrm{Ad}\colon G &\to \mathrm{Gl}(\mathfrak{g})\,, \\ g &\mapsto \mathrm{Ad}_g \equiv (dC_g)\,|_e\,. \end{aligned} \tag{2.34}$$

Hence, the adjoint representation of G assigns to every element $g \in G$ an element of $\mathrm{Gl}\,(\mathfrak{g})$ and therefore is a representation of G on the vector space $\mathfrak{g}$. The adjoint representation ad of $\mathfrak{g}$ on itself can be obtained from the adjoint representation of G as follows:

$$\mathrm{ad}_x(y) = \frac{\partial}{\partial t}\Big|_{t=0} \mathrm{Ad}_{e^{tx}}(y)\,, \qquad x, y \in \mathfrak{g}\,. \tag{2.35}$$

One can show that $\mathrm{ad}_x(y) = [x, y]$, $\forall\, x, y \in \mathfrak{g}$.

2.1.3 Lie group actions on a manifold

In this section we consider the left action of a Lie group G on a differentiable manifold $\mathcal{M}$. Right actions are defined similarly.

Definition 2.1.22. A left action of a Lie group G on a manifold $\mathcal{M}$ is a differentiable map:

$$\begin{aligned} \Phi\colon\ G \times \mathcal{M} &\to \mathcal{M} \\ (p, g) &\mapsto \Phi(g, p)\,, \end{aligned} \tag{2.36}$$

satisiedfying the following conditions:

- For every $p \in \mathcal{M}$ we have $\Phi(e, p) = p$.
- For every $g_1, g_2 \in G$ and for every $p \in \mathcal{M}$ we have $\Phi(g_2, \Phi(g_1, p)) = \Phi(g_2 g_1, p)$.

In order to simplify the notation we will sometimes denote $\Phi(g, p) = g \cdot p$, where $p \in \mathcal{M}$ and $g \in G$.

Remark 2.1.23. For $g \in G$ fixed, we denote by $\Phi_g\colon \mathcal{M} \to \mathcal{M}$ the map defined as:

$$\begin{aligned} \Phi_g\colon \mathcal{M} &\to \mathcal{M}\,, \\ p &\to \Phi_g(p) = \Phi(g, p)\,. \end{aligned} \tag{2.37}$$

For each $g \in G$, Φ_g is a diffeomorphism with inverse given by $\Phi_g^{-1} = \Phi_{g^{-1}}$.

Example 2.1.24.

The map $\Phi\colon G \times G \to G$ given by $l(g,h) = L_g(h)$ is an example of left action of G onto itself. Given a left action $\Phi\colon G \times \mathcal{M} \to \mathcal{M}$ and a fixed point $p \in \mathcal{M}$,

Definition 2.1.25. Let $\Phi\colon G \to \mathcal{M} \to \mathcal{M}$ be a left group action. The isotropy group of Φ on p is defined to be

$$G_p = \{g \in G : g \cdot p = p\} \,. \tag{2.38}$$

It can be seen that G_p is a closed subgroup of G and hence it is a Lie subgroup of G by Cartan's theorem. In addition, for every other point $q \in \mathcal{M}$, G_q and G_p are conjugate to each other in G. Let $\Phi\colon G \times \mathcal{M} \to \mathcal{M}$ be a left group action.

Definition 2.1.26. The orbit of G through $p \in \mathcal{M}$ is $O_p = \{\Phi(g,p) \mid g \in G\}$.

Definition 2.1.27. An action $\Phi\colon G \times \mathcal{M} \to \mathcal{M}$ is said to be:

- **transitive** if $O_p = \mathcal{M}\,,\ \forall\, p \in \mathcal{M}$.
- **free** if G_p is trivial $\forall\, p \in \mathcal{M}$.
- **locally free** if G_p is discrete $\forall\, p \in \mathcal{M}$.
- **effective** if for each $g \in G$ there exists a $p \in \mathcal{M}$ such that $g \cdot p \neq p$.

Definition 2.1.28. Let $\Phi\colon G \times \mathcal{M} \to \mathcal{M}$ be a left group action. If the differentiable map $P\colon G \times \mathcal{M} \to \mathcal{M} \times \mathcal{M}$ defined by $(g,p) \to (\Phi(g,p),p)$ is proper, the action is said to be proper[9].

Definition 2.1.29. Let $\mathfrak{g}$ be a Lie algebra. A Lie algebra action of $\mathfrak{g}$ on a smooth manifold $\mathcal{M}$ is a smooth vector bundle map:

$$\begin{aligned} \mathfrak{g} \times \mathcal{M} &\to T\mathcal{M}\,, \\ (x,p) &\mapsto (p, v_{x,p})\,, \end{aligned} \tag{2.39}$$

such that the associated map $\mathfrak{g} \to \mathfrak{X}(\mathcal{M})$ given by $\xi \mapsto v_x$, where $v_x|_p = v_{x,p}$, is a Lie-algebra homomorphism.

Proposition 2.1.30. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. For any action $\Phi\colon G \times \mathcal{M} \to \mathcal{M}$ of a Lie group G on a manifold $\mathcal{M}$ the induced map:*

$$\begin{aligned} \mathfrak{g} &\to \mathfrak{X}(\mathcal{M})\,, \\ x &\mapsto v_x\,, \end{aligned} \tag{2.40}$$

where:

$$v_x|_p = \frac{d}{dt}\left(e^{-tx} \cdot p\right)|_{t=0}\,, \tag{2.41}$$

is a Lie algebra action of $\mathfrak{g}$ on $\mathcal{M}$. In addition, for $g \in G$ we have:

$$g_* v_x = v_{\mathrm{Ad}_g x}\,. \tag{2.42}$$

If G is simply connected and $\mathcal{M}$ is compact, then the converse is also true, and every $\mathfrak{g}$-action on $\mathcal{M}$ integrates to a G-action.

[9]A map of topological spaces is said to be proper if inverse images of compact subsets are compact.

Example 2.1.31. If v is a complete vector field on $\mathcal{M}$, then

$$\begin{aligned} \rho: \ \mathbb{R} \ &\to \ \mathrm{Diff}\,(\mathcal{M}) \\ t \ &\mapsto \ \rho_t \,, \end{aligned} \tag{2.43}$$

is a smooth action of $\mathbb{R}$ on $\mathcal{M}$.

Let $\sim$ be the orbit equivalence relation on $\mathcal{M}$ defined by

$$p \sim q \ \Leftrightarrow \ p, q \in O_p \,. \tag{2.44}$$

The space of orbits $\mathcal{M}/\sim := \mathcal{M}/G$ is called the orbit space of the G-action Φ on $\mathcal{M}$. The orbit space $\mathcal{M}/G$ can be a very sick space, and in general it will not be a manifold. In fact, it may not be even a Hausdorff topological space. However, there are some simple conditions which guarantee that M/G can be equipped with a unique smooth structure such that the canonical map $\pi\colon \mathcal{M} \to \mathcal{M}/G$ is a smooth map of manifolds. Let then:

$$\begin{aligned} \pi: \ \mathcal{M} \ &\to \ \mathcal{M}/G \\ p \ &\mapsto \ G_p \,. \end{aligned} \tag{2.45}$$

be the canonical projection. We can equip $\mathcal{M}/G$ with the weakest topology for which π is continuous, namely, $\mathcal{U} \subseteq \mathcal{M}/G$ is open if and only if $\pi^{-1}(\mathcal{U})$ is open in $\mathcal{M}$. This is the so-called *quotient topology.*

Proposition 2.1.32. *If $\Phi\colon G \times \mathcal{M} \to \mathcal{M}$ is free and proper then $\mathcal{M}/G$ is Hausdorff.*

Theorem 2.1.33. *Let $\Phi\colon G \times \mathcal{M} \to \mathcal{M}$ be a free and proper group action. Then, there exists a unique smooth structure on $\mathcal{M}/G$ such that:*

- *the induced topology is the quotient topology and $\mathcal{M}/G$ is a smooth manifold.*
- *the projection $\pi\colon G \to \mathcal{M}/G$ is a submersion.*
- $\dim \mathcal{M}/G = \dim \mathcal{M} - \dim G$.

Remark 2.1.34. If G is compact, every smooth action $\Phi\colon G \times \mathcal{M} \to \mathcal{M}$ is proper.

2.2 Locally trivial differentiable fiber bundles

In this section we are going to introduce the concept of locally trivial differentiable fiber bundle. Differentiable fiber bundles are in particular manifolds, i.e., they are manifolds equipped with a very particular extra-structure. We start with the definition of locally trivial differentiable fiber bundle, to which we will refer simply as a fiber bundle.

Definition 2.2.1. Let $\mathcal{F}$, $\mathcal{M}$ and $\mathcal{E}$ be differentiable manifolds and let $\pi : \mathcal{E} \to \mathcal{M}$ be a differentiable surjective map. The quadruple $(\mathcal{E}, \pi, \mathcal{M}, \mathcal{F})$ is a locally trivial differentiable fibre bundle if for every $p \in \mathcal{M}$ there is an open set $\mathcal{U}$ containing p and a diffeomorphism $\phi : \pi^{-1}(\mathcal{U}) \to \mathcal{U} \times \mathcal{F}$ such that the following diagram commutes

$$
\begin{tikzcd}
\pi^{-1}(\mathcal{U}) \arrow[rr, "\phi"] \arrow[drr, "\pi"'] & & \mathcal{U} \times F \arrow[d, "\mathrm{pr}_1"] \\
& & \mathcal{U}
\end{tikzcd}
$$

where pr_1 is the proyection on the first factor. $\mathcal{E}$ is the total space, $\mathcal{M}$ is the base space, $\mathcal{F}$ is the typical fibre and π is the bundle projection. For each $p \in \mathcal{M}$, the set $\mathcal{E}_p \equiv \pi^{-1}(p)$ is the fibre over p, which is diffeomorphic to $\mathcal{F}$. The maps $\phi : \pi^{-1}(\mathcal{U}) \to \mathcal{U} \times \mathcal{F}$ are called the local trivializations of the bundle. Such a local trivialization must be of the form $\phi = \left(\pi_{\pi^{-1}(\mathcal{U})}, \Phi\right)$ where

$$\Phi : \pi^{-1}(\mathcal{U}) \to \mathcal{F}\,, \tag{2.46}$$

is a differentiable map such that

$$\Phi_{|\,\mathcal{E}_p} : \mathcal{E}_p \to \mathcal{F}\,, \tag{2.47}$$

is a diffeomorphism. The pair $(\mathcal{U}, \phi)$, where ϕ is a local trivialization over the open set $\mathcal{U} \subset \mathcal{M}$ is called a bundle chart. A family $(\mathcal{U}_\alpha, \phi_\alpha)_{\alpha \in I}$ such that $(\mathcal{U}_\alpha)_{\alpha \in I}$ is a cover of $\mathcal{M}$ is a bundle atlas. Given two different bundle charts $(\mathcal{U}_\alpha, \phi_\alpha)$ and $(\mathcal{U}_\beta, \phi_\beta)$ such that $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \neq \emptyset$ we have the overlap map

$$\phi_\alpha \circ \phi_\beta^{-1} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \times \mathcal{F} \to \mathcal{U}_\alpha \cap \mathcal{U}_\beta \times \mathcal{F}\,, \tag{2.48}$$

which can be written as follows

$$\phi_\alpha \circ \phi_\beta^{-1}(p, q) = (p, \Phi_{\alpha\beta}(p)(q))\,, \qquad p \in \mathcal{M}\,, \qquad q \in \mathcal{F}\,, \tag{2.49}$$

where $\Phi_{\alpha\beta} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to \mathrm{Diff}\,(\mathcal{F})$ is given by

$$p \mapsto \Phi_{\alpha\beta}(p) = \Phi_{\alpha|\,\mathcal{E}_p} \circ \Phi^{-1}_{\beta|\,\mathcal{E}_p}\,. \tag{2.50}$$

The functions $\Phi_{\alpha\beta}$ are called the transition maps, and satisfy

- $\Phi_{\alpha\alpha}(p) = \mathrm{Id}_{\mathrm{Diff}(\mathcal{F})}\,, \qquad p \in \mathcal{U}_\alpha\,,$
- $\Phi_{\alpha\beta}(p) = \Phi_{\beta\alpha}(p)^{-1}\,, \qquad p \in \mathcal{U}_\alpha \cap \mathcal{U}_\beta\,,$
- $\Phi_{\alpha\beta}(p) \circ \Phi_{\beta\gamma}(p) \circ \Phi_{\gamma\alpha}(p) = \mathrm{Id}_{\mathrm{Diff}(\mathcal{F})}\,, \qquad p \in \mathcal{U}_\alpha \cap \mathcal{U}_\beta \cap \mathcal{U}_\gamma\,,$

for all $\alpha\,, \beta\,, \gamma \in I$. The characterization just given of the transition maps as a map to the diffeomorphisms of $\mathcal{F}$ can be usually restricted to a map onto a lie group G acting on $\mathcal{F}$ by a particular action $\Psi : G \times \mathcal{F} \to \mathcal{F}$. The reader is invited to consult [54, 58] for more details.

Definition 2.2.2. Let $\xi_1 = (\mathcal{E}_1, \pi_1, \mathcal{M}_1, \mathcal{F}_1)$ and $\xi_2 = (\mathcal{E}_2, \pi_2, \mathcal{M}_2, \mathcal{F}_2)$ differentiable fibre bundles. A morphism from ξ_1 to ξ_2 is a couple of maps $F : \mathcal{E}_1 \to \mathcal{E}_2$ and $f : \mathcal{M}_1 \to \mathcal{M}_2$ such that the following diagram commutes

$$\begin{tikzcd}
\mathcal{E}_1 \arrow[r, "F"] \arrow[d, "\pi_1"'] & \mathcal{E}_2 \arrow[d, "\pi_2"] \\
\mathcal{M}_1 \arrow[r, "f"] & \mathcal{M}_2
\end{tikzcd}$$

If F and f are diffeomorphisms, then $(F, f) : \xi_1 \to \xi_2$ is a bundle isomorphism.

Definition 2.2.3. A differentiable global section of a fibre bundle $\xi = (\mathcal{E}, \pi, \mathcal{M}, \mathcal{F})$ is a differentiable map $\sigma : \mathcal{M} \to \mathcal{E}$ such that $\pi \circ \sigma = \mathrm{Id}_{\mathcal{M}}$. A differentiable local section over an open set $\mathcal{U}$ is a differentiable map $\sigma : \mathcal{M} \to \mathcal{U}$ such that $\pi \circ \sigma = \mathrm{Id}_{\mathcal{U}}$.
The set of differentiable sections of ξ is denoted by $\Gamma(\xi)$ or $\Gamma(\mathcal{E})$. Notice that a fibre bundle may not have any global section.

Definition 2.2.4. Let V be a finite dimensional vector space over the complex or real numbers. A smooth vector bundle with typical fibre V is a fibre bundle $(\mathcal{E}, \pi, \mathcal{M}, V)$ such that

- for each $p \in \mathcal{M}$ we have that $\mathcal{E}_p = \pi^{-1}(p)$ is a vector space isomorphic to V.
- for every $p \in \mathcal{M}$ there exist a bundle chart $(\mathcal{U}_\alpha, \phi_\alpha)$ containing p such that

$$\Phi_{|\,\mathcal{E}_p} : \mathcal{E}_p \to V\,, \tag{2.51}$$

  is a vector space isomorphism, where $\phi = \left(\pi_{\pi^{-1}(\mathcal{U})}, \Phi\right)$.

The typical example of vector bundle is the tangent bundle $T\mathcal{M}$ over a manifold $\mathcal{M}$. The notion of bundle morphism, given in definition (2.2.2) specializes to vector bundles by requiring $F_{|\,\pi_1^{-1}(p)} : \pi_1^{-1}(p) \to \pi_2^{-1}(f(p))$ to be linear.

Given two vector bundles $\pi_1 : \mathcal{E}_1 \to \mathcal{M}$ and $\pi_2 : \mathcal{E}_2 \to \mathcal{M}$, we can define the *Whitney sum bundle* $\pi_1 \oplus \pi_2 : \mathcal{E}_1 \oplus \mathcal{E}_2 \to \mathcal{M}$ such that the fibre at a point $p \in \mathcal{M}$ is given by $(\mathcal{E}_1 \oplus \mathcal{E}_2)_p = \mathcal{E}_{1\,p} \oplus \mathcal{E}_{2\,p}$.

The pull-back of a vector bundle $\pi : \mathcal{E} \to \mathcal{M}$ by a smooth map $f : \mathcal{N} \to \mathcal{M}$, where $\mathcal{N}$ is a differentiable manifold, is the vector bundle $(f^*\mathcal{E})$ over $\mathcal{N}$ defined as follows

$$f^*\mathcal{E} = \{(q, e) \in \mathcal{N} \times \mathcal{E} \mid f(q) = \pi(e)\} \subset \mathcal{N} \times \mathcal{E}\,, \tag{2.52}$$

and equipped with the subspace topology and the projection map $\mathrm{pr}_1 : f^*\mathcal{E} \to \mathcal{N}$ given by the projection onto the first factor

$$\mathrm{pr}_1(q, e) = q\,. \tag{2.53}$$

Notice that the following diagram commutes

$$\begin{tikzcd} f^*\mathcal{E} \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & \mathcal{E} \arrow[d, "\pi"] \\ \mathcal{N} \arrow[r, "f"] & \mathcal{M} \end{tikzcd}$$

where pr_2 is the projection on the second factor. If $(\mathcal{U}, \phi)$ is a local trivialization of $\mathcal{E}$, then $\left(f^{-1}(\mathcal{U}), \psi\right)$ is a local trivialization of $f^*\mathcal{E}$ where

$$\psi(q,e) = (q, \mathrm{pr}_2(\phi(e))) , \qquad \forall\, (q,e) \in f^*\mathcal{E} . \tag{2.54}$$

Therefore, the fibre at a point $q \in \mathcal{N}$ is given by

$$(f^*\mathcal{E})_q = \mathcal{E}_{f(q)} . \tag{2.55}$$

A section $\sigma \in \Gamma(\mathcal{E})$ induces a section $f^*\sigma \in \Gamma(f^*\mathcal{E})$ defined by $f^*\sigma = \sigma \circ f$.

Example 2.2.5. As an example of pull-back of a vector bundle we are going to consider the pull-back of the tangent bundle $T\mathcal{M}$ of a differentiable manifold $\mathcal{M}$. Let $\mathcal{N}$ be a differentiable manifold and let

$$f : \mathcal{N} \to \mathcal{M} \tag{2.56}$$

be a map. The pull-back bundle is defined as follows

$$f^*T\mathcal{M} = \{(q,e) \in \mathcal{N} \times T\mathcal{M} \mid f(q) = \pi(e)\} \subset \mathcal{N} \times T\mathcal{M} . \tag{2.57}$$

Notice that the following diagram commutes

$$\begin{tikzcd} f^*T\mathcal{M} \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & T\mathcal{M} \arrow[d, "\pi"] \\ \mathcal{N} \arrow[r, "f"] & \mathcal{M} \end{tikzcd}$$

Notice that in general $f^*T\mathcal{M}$ is not equal to $T\mathcal{N}$. Only when f is a diffeomorphism we have $f^*T\mathcal{M} \simeq T\mathcal{N}$.

2.3 Courant algebroids

In this section we consider a particular type of vector bundle called Courant algebroid. A Courant algebroid is, roughly speaking, an extension of the tangent bundle of a smooth manifold M by means of an extrinsic vector bundle $E \to M$ equipped with a non-degenerate symmetric bilinear form and a bracket satisfying a particular relaxed version of the standard condition of anitsymmetricity, Leibniz rule and Jacobi identity.

The canonical example of Courant algebroid was first introduced by T. Courant in reference [59] in order to obtain a unified description of pre-symplectic and Poisson structures in Dirac's theory of constrained mechanical systems. Courant algebroids were then abstractly defined for the first time by Liu, Weinstein and Xu in reference [60], and by know there are several equivalent definitions of Courant algebroids available in the literature. Here we will use the definition given in reference [61] by Ševera:

Definition 2.3.1 ([61]). A **Courant algebroid** $(E, \langle\cdot,\cdot\rangle, [\cdot,\cdot], \pi)$ over a manifold M consists of a vector bundle $E \to M$ together with a nondegenerate symmetric bilinear form $\langle\cdot,\cdot\rangle$ on E, a (Dorfman) bracket $[\cdot,\cdot]$ on the sections $\Gamma(E)$, and a bundle map $\pi : E \to TM$ such that the following properties are satisfied, for $e_1, e_2, e_3 \in \Gamma(E)$ and $\phi \in C^\infty(M)$:

(C1): $[e_1, [e_2, e_3]] = [[e_1, e_2], e_3] + [e_2, [e_1, e_3]]$,

(C2): $\pi([e_1, e_2]) = [\pi(e_1), \pi(e_2)]$,

(C3): $[e_1, \phi e_2] = \pi(e_1)(\phi)e_2 + \phi[e_1, e_2]$,

(C4): $\pi(e_1)\langle e_2, e_3\rangle = \langle[e_1, e_2], e_3\rangle + \langle e_2, [e_1, e_3]\rangle$,

(C5): $[e_1, e_2] + [e_2, e_1] = \pi^* d\langle e_1, e_2\rangle$.

The map $\pi\colon E \to TM$ is usually called the *anchor map*. Notice that given an Courant algebroid E, we can always identify $E^* \simeq E$ by using the bilinear $\langle\cdot,\cdot\rangle$ and hence we obtain a map:

$$\pi^*\colon T^*M \to E\,, \tag{2.58}$$

dual to $\pi\colon E \to TM$. This is the map appearing in item $C5$ of definition 2.3.1. The bracket in Definition 2.3.1 goes under the name of Dorfman bracket $[\cdot,\cdot]$. It satisfies the Jacobi identity, namely item C1, but fails to be antisymmetric, and relates to the skew-symmetrized *Courant bracket* $[\![\cdot,\cdot]\!]$, by

$$[\cdot,\cdot] = [\![\cdot,\cdot]\!] + \pi^* d\langle\cdot,\cdot\rangle\,. \tag{2.59}$$

The definition in the original reference [60], differs from definition 2.3.1 in the bracket used (see also [62]). An explicit example of Courant algebroid is now in order.

Example 2.3.2. The simplest example of Courant algebroid is the **standard Courant algebroid** $E = TM \oplus T^*M$ over a manifold M, equipped with the **standard Dorfman bracket**:

$$[v_1 + \alpha_1, v_2 + \alpha_2] = [v_1, v_2]_L + \mathcal{L}_{v_1}\alpha_2 - \iota_{v_2} d\alpha_1\,, \qquad v_1, v_2 \in \mathfrak{X}(M)\,, \qquad \alpha_1, \alpha_2 \in \Omega^1(M)\,, \tag{2.60}$$

and the **standard symmetric pairing**:

$$\langle v_1 + \alpha_1, v_2 + \alpha_2\rangle = \frac{1}{2}(\iota_{v_1}\alpha_2 + \iota_{v_2}\alpha_1)\,, \tag{2.61}$$

where $[\cdot,\cdot]_L$ denotes the standard Lie bracket on $\mathfrak{X}(M)$. The anchor map $\pi\colon E \to TM$ is simply the obvious projection on the tangent bundle.

It was noticed in reference [61] that one can twist the standard Dorfman bracket by using a closed three-form H as follows:

$$[v_1 + \alpha_1, v_2 + \alpha_2]_H = [v_1, v_2]_L + \mathcal{L}_{v_1}\alpha_2 - \iota_{v_2} d\alpha_1 + \iota_{v_1}\iota_{v_2} H\,, \tag{2.62}$$

and still obtain a Courant algebroid in $TM \oplus T^*M$, with the same anchor and symmetric product. This way, it is obtained the so-called H-twisted standard Courant algebroid. The standard Courant algebroid is, as we will see in a moment, the prototype of an exact Courant algebroid.

Definition 2.3.3. [61] A Courant algebroid $(E, \langle\cdot,\cdot\rangle, [\cdot,\cdot], \pi)$ over M is exact if and only if the following sequence of vector bundles

$$0 \to T^*M \xrightarrow{\pi^*} E \xrightarrow{\pi} TM \to 0\,, \tag{2.63}$$

is exact.

Definition 2.3.4. [61] A splitting of an exact Courant algebroid $(E, \langle\cdot,\cdot\rangle, [\cdot,\cdot], \pi)$ over a manifold M is a map of vector bundles $s\colon TM \to E$ such that

1. $\pi \circ s = \mathbb{I}_{TM}$,
2. $\langle s(v_1), s(v_2)\rangle = 0$ for all $v_1, v_2 \in \mathfrak{X}(M)$.

Definition 2.3.4 means that a splitting of an exact Courant algebroid is an isotropic splitting of the sequence of vector bundles 2.63. Notice that $\pi^*(T^*M) \cap s(TM) = \{0\}$. The exactness condition in the definition 2.3.3 forces $\pi^*(T^*M)$ to be isotropic in E and thus the symmetric pairing $\langle\cdot,\cdot\rangle$ is bound to be of split signature. If s is an splitting, then for every two-form $b \in \Omega^2(M)$ we can construct another splitting s' as follows

$$s'(v) = s(v) + \frac{1}{2}\pi^* b(v)\,, \tag{2.64}$$

and in fact every two splittings of a Courant algebroid differ by a two-form on M in this way [63]. In other words, the space of splittings of a Courant algebroid is an affine space modeled on $\Omega^2(M)$. Given an exact Courant algebroid $(E, \langle\cdot,\cdot\rangle, [\cdot,\cdot], \pi)$, any isotropic splitting $s\colon TM \to E$, has an associated *three-form curvature*:

$$H(v_1, v_2, v_3) = \langle [\![s(v_1), s(v_2)]\!], s(v_3)\rangle\,, \qquad v_1, v_2, v_3 \in \mathfrak{X}(M)\,. \tag{2.65}$$

It can be proven that given another splitting s' then the corresponding three-form:

$$H'(v_1, v_2, v_3) = \langle [\![s'(v_1), s'(v_2)]\!], s'(v_3)\rangle\,, \qquad v_1, v_2, v_3 \in \mathfrak{X}(M)\,. \tag{2.66}$$

is related to H as follows:

$$H' = H + db\,, \tag{2.67}$$

where $s' - s = b \in \Omega^2(M)$. As observed first by Ševera [61], given an exact Courant algebroid $(E, \langle\cdot,\cdot\rangle, [\cdot,\cdot], \pi)$, the class $[H] \in H^3(M)$ does not depend on the splitting. It is called the Ševera class of the exact Courant algebroid and its importance steams from the fact that it classifies exact Courant algebroids up to isomorphism. In other words, two exact Courant algebroids are isomorphic if and only if they have the same Ševera class.

Notice that for an exact Courant algebroid $(E, \langle\cdot,\cdot\rangle, [\cdot,\cdot], \pi)$, any isotropic splitting $s\colon TM \to E$ determines an isomorphism

$$s + \frac{1}{2}\pi^*\colon TM \oplus T^*M \to E,$$

and the trasported bracket and pairing are given by 2.62 and 2.61, respectively. Therefore, exact Courant algebroids over a manifold M can be always modeled by the corresponding generalized tangent bundle $TM \oplus T^*M$ equipped with the standard symmetric pairing and the H-twisted Courant bracket 2.62. Exact Courant algebroids are intimately related to two-plectic manifolds (see section 6 for more details) as they are canonically equipped with a closed three-form, its Ševera class.

Chapter 3

Symplectic Geometry

Symplectic geometry/topology is a classical and well-established branch of differential geometry/topology, with its roots and original motivation lying in the mathematical description of classical mechanical systems as well as in their quantization. Symplectic manifolds are also interesting mathematical objects by themselves, and play a crucial role in many different areas of physics and mathematics. Symplectic geometry is nowadays a very active research field, with applications extending through numerous branches of geometry, topology and theoretical physics. The goal of this chapter is to introduce the concept of (co)moment map [64, 65] and symplectic reduction [66]. References for this chapter include [5, 6, 67].

3.1 Symplectic vector spaces

The simplest type of symplectic manifold is a symplectic vector space. In addition, the tangent space of every symplectic manifold is canonically a symplectic vector space. Therefore, it is reasonable to consider in some detail symplectic vector spaces before dealing with general symplectic manifolds.

Definition 3.1.1. Let V be a vector space. The pair (V, ω) is a symplectic vector space if $\omega \in \Lambda^2 V^*$ is non-degenerate, that is, if the kernel:

$$\ker \omega \equiv \{v \in V \mid \omega(v, w) = 0\,,\ \forall w \in V\}\,, \tag{3.1}$$

is trivial.

Remark 3.1.2. Symplectic vector spaces must be even-dimensional, since a symplectic form on an odd-dimensional vector space necessarily has a kernel.

A morphism of symplectic vector spaces $(V_1,\, \omega_1)$ and $(V_1,\, \omega_1)$ is a linear map $F\colon V_1 \to V_2$ that preserve the corresponding symplectic structures, namely that satisfy $F^*\omega_2 = \omega_1$. Two symplectic vector spaces $(V_1,\, \omega_1)$ and $(V_1,\, \omega_1)$ are said to be *symplectomorphic* if there exists an injective and surjective morphism between them.

Example 3.1.3. The canonical example of symplectic vector space consists of $\mathbb{R}^{2n}$, for some $n \in \mathbb{N}$, with basis $\{e_1, \dots, e_n, f_1, \dots, f_n\}$ equipped with the bilinear form ω given by:

$$\omega_0\,(e_i, e_j) = 0\,, \quad \omega_0\,(f_i, f_j) = 0\,, \quad \omega_0\,(e_i, f_j) = -\omega_0\,(f_j, e_i) = \delta_{ij}\,. \tag{3.2}$$

The two-form $\omega \in \Lambda^2(\mathbb{R}^{2n})^*$ is a symplectic structure on $\mathbb{R}^{2n}$. Every symplectic vector space is non-canonically symplectomorphic to $\left(\mathbb{R}^{2n}, \omega\right)$ for the appropriate n.

Example 3.1.4. Let E be a complex vector space of dimension n, equipped with a complex, positive definite inner product $h : E \times E \to \mathbb{C}$. Then E, taken as a real vector space, equipped with the bilinear form $\omega = \Im\mathrm{m}\,(h)$, is a symplectic vector space. The condition $h\,(v_1, v_2) = \overline{h\,(v_2, v_1)}$ translates into the antisymmetry of $\omega = \Im\mathrm{m}\,(h)$ as a real form on V.

Definition 3.1.5. Let (V, ω) be a symplectic vector space. The **symplectic complement** of a linear subspace $W \subset V$ with respect ω is defined to be:

$$W^{\omega} = \{v \in V \mid \omega(v, w) = 0\,,\ \forall w \in W\}\,. \tag{3.3}$$

Remark 3.1.6. The symplectic complement W^{ω} need not be transversal to W.

Definition 3.1.7. Let $W \subset V$ a vector subspace of a symplectic vector space (V, ω).

- W is isotropic if $W \subset W^{\omega}$.
- W is coisotropic if $W^{\omega} \subset W$.
- W is symplectic if $W \cap W^{\omega} = \{0\}$.
- W is Lagrangian if $W = W^{\omega}$.

Remark 3.1.8. We have that W is isotropic if and only if $\omega|_W = 0$. In addition W is symplectic if and only if $\omega|_W$ is non-degenerate.

Lemma 3.1.9. *Let (V, ω) be a symplectic vector space. Then, for any vector subspace $W \subset V$ we have:*

$$(W^{\omega})^{\omega} = W\,, \qquad \dim W + \dim W^{\omega} = \dim V\,. \tag{3.4}$$

Proof. The symplectic form ω defines an isomorphism of vector spaces $\tilde{\omega}\colon V \to V^*$ given by $v \mapsto \iota_v \omega$. It follows that $\tilde{\omega}(W^{\omega}) = W^{\perp}$, where $W^{\perp}$ denotes the annihilator of W in V^*. The result follows. $\square$

The following is the main theorem on symplectic vector spaces.

Theorem 3.1.10. *Let (V, ω) be a symplectic vector space of dimension $2n$. Then, there exists a basis $(e_1, \dots, e_n, f_1, \dots, f_n)$ of V such that:*

$$\omega(e_i, e_j) = 0\,, \quad \omega(f_i, f_j) = 0\,, \quad \omega(e_i, f_j) = -\omega(f_j, e_i) = \delta_{ij}\,. \tag{3.5}$$

Such basis is called a **symplectic basis**. *Moreover, there exists a symplectomorphism from (V, ω) to $(\mathbb{R}^{2n}, \omega_0)$.*

Proposition 3.1.11. *A two-form ω on V is non-degenerate if and only if ω^n is non-zero.*

Proof. If ω is non-degenerate then clearly $\omega^n \neq 0$. Let now assume that ω^n is non-degenerate. Then, given a $v \in V$ we have $\iota_v \omega^n = n \iota_v \omega \wedge \omega^{n-1} \neq 0$ and the result follows. $\square$

Lemma 3.1.12. *Any isotropic subspace $W \subset V$ is contained in a Lagrangian subspace. Moreover, any basis of a Lagrangian subspace can be extended to a symplectic basis of (V, ω).*

The quotient of every coisotropic subspace $W \subset V$ by its symplectic complement canonically yields a new symplectic vector space. This construction is usually called *symplectic reduction*.

Lemma 3.1.13. *Let (V, ω) be a symplectic vector space and let $W \subset V$ be a coisotropic subspace. Then the following hold:*

- *The quotient $\tilde{V} \equiv W/W^{\omega}$ carries a canonical symplectic structure induced by ω.*

- *If $L \subset V$ is a Lagrangian subspace of V, then:*

$$\tilde{L} = \left(L \cap W \oplus W^{\omega}\right)/W^{\omega}\,, \tag{3.6}$$

is a Lagrangian subspace of $\tilde{V}$.

Given a symplectic vector space $(V,\,\omega)$, the group of automorphisms that preserve ω is denoted by $\mathrm{Sp}\,(V)$, which is a closed Lie subgroup of $\mathrm{Gl}\,(V)$, the group of automorphisms of V. Since every symplectic vector space (V,ω) is symplectomorphic to $(\mathbb{R}^{2n},\omega_0)$, it is enough to consider the later case, which shows that:

$$\mathrm{Sp}(V) \simeq \mathrm{Sp}(2n,\mathbb{R}) \simeq \left\{A \in \mathrm{Gl}(\mathbb{R}^{2n}) \simeq \mathrm{Gl}(2n,\mathbb{R}) \mid A^*\omega_0 = \omega_0\right\}\,. \tag{3.7}$$

In particular, equation (3.7) clearly shows that $\mathrm{Sp}(2n,\mathbb{R})$ is a closed Lie subgroup of $\mathrm{Gl}(2n,\mathbb{R})$. Let J_0 denote the complex structure on $\mathbb{R}^{2n}$ associated to ω_0 by means of the standard euclidean metric on $\mathbb{R}^{2n}$. The complex structure J_0 on $\mathbb{R}^{2n}$ allows for the following standard identification:

$$(\mathbb{R}^{2n},J_0) \simeq (\mathbb{C}^n,i)\,. \tag{3.8}$$

Let us denote by $\mathcal{N}(V) \subset \Lambda^2 V^*$ the subspace of non-generate two forms in $\Lambda^2 V^*$. The fact that every symplectic vector space is symplectomorphic to the canonical example $(\mathbb{R}^{2n},\omega_0)$ implies that $\mathrm{Gl}(V)$ acts transitively on $\mathcal{N}(V)$. Since, every non-degenerate two-form is stabilized by $\mathrm{Sp}(V)$, we obtain the following diffeomorphism:

$$\mathcal{N}(V) \simeq \frac{\mathrm{Gl}(V)}{\mathrm{Sp}(V)}\,. \tag{3.9}$$

We have:

$$\dim\,\mathrm{Sp}(V) = \frac{n(n+1)}{2}\,,\qquad \dim\,\mathcal{N}(V) = \dim\,\mathrm{Gl}(V) - \dim\,\mathrm{Sp}(V) = \frac{n(n-1)}{2}\,, \tag{3.10}$$

and hence $\mathcal{N}(V)$ is of the same dimension of $\Lambda^2 V^*$. In particular, $\mathcal{N}(V)$ is open in $\Lambda^2 V^*$. The Lie algebra $\mathfrak{sp}(V)$ of $\mathrm{Sp}(V)$ is given by the endomorphisms $A \in \mathfrak{gl}(V)$ such that:

$$\omega(A\,\cdot,\cdot) + \omega(\cdot, A\,\cdot) = 0\,. \tag{3.11}$$

3.2 Symplectic manifolds

Definition 3.2.1. A symplectic manifold $(\mathcal{M},\omega)$ is a real manifold equipped with a smooth, point-wise non-degenerate , global section ω of $\Lambda^2 T^*\mathcal{M}$. Therefore, at every point $p \in \mathcal{M}$, $T_p\mathcal{M}$ is a symplectic vector space equipped with the symplectic form $\omega|_p$.

Given two symplectic manifolds $(M_1,\,\omega_1)$ and $(M_2,\,\omega_2)$, a symplectomorphism is a diffeomorphism $F : \mathcal{M}_1 \to \mathcal{M}_2$ such that $F^*\omega_2 = \omega_1$. The group of symplectomorphisms of a symplectic manifold (M,ω) onto itself is denoted by $\mathrm{Symp}\,(\mathcal{M},\,\omega)$. The non-degeneracy of ω has very important consequences on the geometry and topology of a symplectic manifold $(\mathcal{M},\omega)$, for instance:

- Every symplectic manifold is orientable, with volume form given by the Liouville form

$$\Lambda_L = \frac{\omega^n}{n!} \neq 0\,, \tag{3.12}$$

where $2n$ is the dimension of $\mathcal{M}$.

- Every symplectic manifold is almost-complex, namely it admits almost-complex structures.
- There exists a canonical isomorphism between the vector fields and one-forms on $\mathcal{M}$, given by

$$\begin{array}{rcl}\tilde{\omega}:\mathfrak{X}(\mathcal{M}) & \to & \Omega(\mathcal{M}) \\ v & \mapsto & \iota_v\omega\,. \end{array} \tag{3.13}$$

Example 3.2.2. The simplest example of symplectic manifold is a Riemann surface equipped with its volume form, which is clearly non-degenerate and closed. A symplectomorphism is then just a volume-preserving diffeomorphism.

We denote by $\mathfrak{X}_{\mathrm{Sym}}(\mathcal{M})$ the set of all vector fields in $\mathcal{M}$ that preserves ω, that is

$$\mathcal{L}_v\omega = d\iota_v\omega + \iota_v d\omega = d\iota_v\omega = 0\,, \tag{3.14}$$

where we have used (2.18). An element of $\mathfrak{X}_{\mathrm{Sym}}(\mathcal{M})$ is called a symplectic vector field, and generates symplectomorphisms through the corresponding flow. As already mentioned, due to the non-degeneracy of ω, for every one-form $\xi \in \Omega(\mathcal{M})$ there exists a unique vector field $v \in \mathfrak{X}(\mathcal{M})$ such that:

$$\iota_v\omega = \xi\,. \tag{3.15}$$

In particular, for any function $f \in C^\infty(\mathcal{M})$ there exists a unique vector field $v_f \in \mathfrak{X}(\mathcal{M})$ such that

$$\iota_{v_f}\omega = -df\,. \tag{3.16}$$

ι_{v_f} is the so-called Hamiltonian vector field associated to f. The space of vector fields satisfying equation (3.16) for some function $f \in C^\infty(\mathcal{M})$ is denoted by $\mathfrak{X}_{\mathrm{Ham}}(\mathcal{M})$, the space of hamiltonian vector fields.

Proposition 3.2.3. *Every Hamiltonian vector field is a symplectic vector field. That is*

$$\mathfrak{X}_{\mathrm{Ham}}(\mathcal{M}) \subseteq \mathfrak{X}_{\mathrm{Sym}}(\mathcal{M})\,. \tag{3.17}$$

Proof. Given a Hamiltonian vector field v_f respect to some function $f \in C^\infty(\mathcal{M})$, we have, using Cartan's identity (2.18), $\mathcal{L}_{v_f}\omega = d^2 f = 0$. □

Therefore, from Eq. (3.14) and (3.16) respectively, we see that $\iota_v\omega$ is closed for a symplectic vector field and exact for a hamiltonian vector field. Restricting $\tilde{\omega}$ in Eq. (3.13) to the set of hamiltonian vector fields $\mathfrak{X}_{\mathrm{Ham}}(\mathcal{M})$ we obtain a new isomorphism

$$\omega|_{\mathfrak{X}_{\mathrm{Ham}}} : \mathfrak{X}_{\mathrm{Ham}}(\mathcal{M}) \to B^1(\mathcal{M})\,, \tag{3.18}$$

where $B^1(\mathcal{M}) = \Omega^1(\mathcal{M}) \cap \mathrm{Im}\, d$ is the space of exact one-forms. Similarly, restricting $\tilde{\omega}$ to the set of symplectic vector fields $\mathfrak{X}_{\mathrm{Sym}}(\mathcal{M})$ we obtain another isomorphism

$$\omega|_{\mathfrak{X}_{\mathrm{Sym}}} : \mathfrak{X}_{\mathrm{Ham}}(\mathcal{M}) \to Z^1(\mathcal{M})\,, \tag{3.19}$$

where $Z^1(\mathcal{M}) = \Omega^1(\mathcal{M}) \cap \mathrm{Ker}\, d$ is now the space of closed one-forms. Therefore, the quotient of symplectic and hamiltonian vector fields is just the first de Rham cohomology group:

$$H^1(\mathcal{M}) = \frac{\mathfrak{X}_{\mathrm{Sym}}(\mathcal{M})}{\mathfrak{X}_{\mathrm{Ham}}(\mathcal{M})}\,. \tag{3.20}$$

Therefore, the following exact sequence of vector spaces holds

$$0 \to \mathfrak{X}_{\mathrm{Ham}}(\mathcal{M}) \to \mathfrak{X}_{\mathrm{Sym}}(\mathcal{M}) \to H^1(\mathcal{M}) \to 0\,. \tag{3.21}$$

Hence, if $H^1(\mathcal{M}) = 0$, every symplectic vector field is Hamiltonian.

Proposition 3.2.4. *Given two symplectic vector fields $v_{f_1}, v_{f_2} \in \mathfrak{X}_{\mathrm{Sym}}(\mathcal{M})$ we have that $[v_{f_1}, v_{f_2}]$ is Hamiltonian, with Hamiltonian function $\omega(v_{f_1}, v_{f_2})$.*

Proof. Let $v_{f_1}, v_{f_2} \in \mathfrak{X}_{\mathrm{Sym}}(\mathcal{M})$. Then we have

$$d\omega(v_{f_1}, v_{f_2}) = di_{v_{f_2}} i_{v_{f_1}}\omega = \mathcal{L}_{v_{f_2}} i_{v_{f_1}}\omega - i_{v_{f_2}} di_{v_{f_1}}\omega = i_{\mathcal{L}_{v_{f_2}} v_{f_1}}\omega = -i_{[v_{f_1}, v_{f_2}]}\omega\,. \tag{3.22}$$

□

Therefore, $[\mathfrak{X}_{\mathrm{Sym}}(\mathcal{M}), \mathfrak{X}_{\mathrm{Sym}}(\mathcal{M})] \subseteq \mathfrak{X}_{\mathrm{Ham}}(\mathcal{M})$, and in particular $\mathfrak{X}_{\mathrm{Ham}}(\mathcal{M})$ is an ideal in the Lie algebra $\mathfrak{X}_{\mathrm{Sym}}(\mathcal{M})$, and the quotient Lie algebra is abelian. Hence, 3.21 is an exact sequence of Lie algebras, where $H^1(\mathcal{M})$ carries the trivial Lie algebra structure.

Consider now the following surjective map

$$\begin{array}{rcl} h:\, C^\infty(\mathcal{M}) & \to & \mathfrak{X}_{\mathrm{Ham}}(\mathcal{M}) \\ f & \mapsto & v_f\,. \end{array} \tag{3.23}$$

The kernel of h is the space $Z^0(\mathcal{M}) = H^0(\mathcal{M})$ of locally constant functions. Therefore, we can write the following exact sequence of vector spaces

$$0 \to Z^0(\mathcal{M}) \to C^\infty(\mathcal{M}) \to \mathfrak{X}_{\mathrm{Ham}}(\mathcal{M}) \to 0\,. \tag{3.24}$$

It is possible to define a Lie algebra structure on $C^\infty(\mathcal{M})$ such that (3.24) is an exact sequence of Lie algebras.

Definition 3.2.5. Let $(\mathcal{M}, \omega)$ be a symplectic manifold. The Poisson bracket of two funcions $f, g \in C^\infty(\mathcal{M})$ is defined as

$$\{f, g\} = \omega(v_f, v_g)\,. \tag{3.25}$$

The Poisson bracket is anti-symmetric. Using Cartan's identity (2.18), the Poisson bracket can be rewritten as follows

$$\{f, g\} = \mathcal{L}_{v_f} g = -\mathcal{L}_{v_g} f\,. \tag{3.26}$$

Therefore, if $\{f, g\} = 0$, then f is constant along solution curves of v_g and vice-versa.

Proposition 3.2.6. *The Poisson bracket defines a Lie algebra structure into $C^\infty(\mathcal{M}, \mathbb{R})$. The map*

$$\begin{array}{rcl} C^\infty(\mathcal{M}) & \to & \mathfrak{X}(\mathcal{M}) \\ f & \mapsto & v_f\,, \end{array} \tag{3.27}$$

is a Lie algebra isomorphism, that is

$$v_{\{f,g\}} = [v_f, v_g]\,. \tag{3.28}$$

Proof. We have to prove that the Poisson bracket satisfies the Jacobi identity. This follows from

$$\begin{aligned} \{f,\{g,h\}\} = \mathcal{L}_{v_f}\{g,h\} = \omega\left([v_f,v_g],v_h\right) + \omega\left(v_g,[v_f,v_h]\right) = \\ \omega\left(v_{\{f,g\}},v_h\right) + \omega\left(v_g,v_{\{f,g\}}\right) = \{h,\{f,g\}\} + \{g,\{f,h\}\}\,. \end{aligned} \tag{3.29}$$

Equation (3.28) is a particular instance of proposition 3.2.4. □

Proposition 3.2.7. *The algebra $(C^\infty(\mathcal{M},\mathbb{R}),\{\cdot,\cdot\})$ is a Poisson algebra*[1].

Proof. We have to proof that the Poisson bracket satisfies equation (4.36). Indeed we have

$$\{fg,h\} = -\mathcal{L}_{v_h}(fg) = -\mathcal{L}_{v_h}(f)g - f\mathcal{L}_{v_h}(g) = -\{f,h\}g - f\{g,h\}\,. \tag{3.30}$$

□

We will define now submanifolds of a symplectic manifold $(\mathcal{M},\omega)$ that can be defined in a natural way using the symplectic form ω.

Definition 3.2.8. A submanifold $Q\subset\mathcal{M}$ is is called $\{symplectic, isotropic, coisotropic, Lagrangian\}$ if for every $q\in Q$ the tangent space T_qQ is respectively $\{symplectic, isotropic, coisotropic, Lagrangian\}$.

Example 3.2.9. Given a symplectic manifold $(\mathcal{M},\omega)$ the manifold $\mathcal{M}\times\mathcal{M}$ is a symplectic manifold with symplectic form $-\omega\times\omega$. For every $p\in\mathcal{M}$, the manifold $\mathcal{M}\times p$ or $p\times\mathcal{M}$ is symplectic, whereas the diagonal of $\mathcal{M}\times\mathcal{M}$ is Lagrangian.

As an application of the previous example we have the following result.

Proposition 3.2.10. *Let $(\mathcal{M},\omega)$ be a symplectic manifold and let $f\colon\mathcal{M}\to\mathcal{M}$ be a diffeomorphism. Then f is a symplectomorphism if and only if:*

$$\mathrm{graph}(f) = \{(p,f(p))\,|\,p\in\mathcal{M}\}\subset\mathcal{M}\times\mathcal{M}\,, \tag{3.31}$$

is a Lagrangian submanifold of $(\mathcal{M}\times\mathcal{M},-\omega\times\omega)$.

3.3 Moment maps and symplectic reduction

Let $(\mathcal{M},\omega)$ be a symplectic manifold. In this section we will consider the G-action $\Phi\colon G\times\mathcal{M}\to\mathcal{M}$ of a Lie group G on a symplectic manifold $(\mathcal{M},\omega)$. As usual, for every $g\in G$ we define $\Phi_g := \Phi(g,\cdot)\colon\mathcal{M}\to\mathcal{M}$ and we denote the fundamental vector field associated to the G-action Φ as follows:

$$v_x|_p = \frac{d}{dt}\left(e^{-tx}\cdot p\right)|_{t=0}\,,\qquad x\in\mathfrak{g}\,,\qquad p\in\mathcal{M}\,. \tag{3.32}$$

Definition 3.3.1. The G-action Φ is said to be *symplectic* if the Lie group G acts on $(\mathcal{M},\omega)$ by symplectomorphisms, i.e., if for all $g\in G$ we have:

$$\Phi_g^*\omega = \omega\,. \tag{3.33}$$

Proposition 3.3.2. *Let Φ a symplectic G-action on $(\mathcal{M},\omega)$, and let $f,h\in C^\infty(\mathcal{M})$ functions such that:*

$$\Phi_g^* f = f\,,\qquad \Phi_g^* h = h\,,\qquad \forall\, g\in G\,. \tag{3.34}$$

[1] See definition 4.1.14.

Then $\Phi_g^*\{f,g\} = \{f,g\}$. In addition, let $\mathcal{O}_p$ be the orbit of the G-action passing through $p \in \mathcal{M}$[2] and let $i\colon \mathcal{O} \hookrightarrow \mathcal{M}$ be the canonical inclusion. Then, $i^*\omega$ is a constant-rank two-form on $\mathcal{O}$ invariant under the restricted action $\Phi|_{\mathcal{O}}\colon G\times\mathcal{O}\to\mathcal{O}$.

Proof. Since Φ_g is a diffeomorphism for every $g \in G$, the first statement simply follows from the identity:

$$\Phi_g^*\{f,g\} = \left\{\Phi_g^* f, \Phi_g^* g\right\}. \tag{3.35}$$

To prove the second statement notice that the canonical injection $i\colon \mathcal{O}\hookrightarrow\mathcal{M}$ satisfies:

$$\Phi_g \circ i = (\Phi|_{\mathcal{O}})_g \circ i\,. \tag{3.36}$$

Hence:

$$i^*\omega = i^*\Phi_g^*\omega = (\Phi|_{\mathcal{O}})_g^* i^*\omega\,, \tag{3.37}$$

and we conclude. □

Proposition 3.3.3. *If the G-action Φ on $\mathcal{M}$ is symplectic, the corresponding fundamental vector field v_x, $x\in\mathfrak{g}$ is locally hamiltonian. Conversely, if G is connected and for every $x\in\mathfrak{g}$ the vector field v_x is locally Hamiltonian then the corresponding G-action Φ is symplectic.*

Proof. The flow of v_x is given by:

$$\varphi(t,p) = e^{-tx}\cdot p\,. \tag{3.38}$$

The fact that Φ is symplectic implies that $\Phi_{e^{-tx}}^*\omega = \omega$, which is equivalent to $\mathcal{L}_{v_x}\omega = d\iota_{v_x}\omega = 0$. For the converse, it is enough to notice that since G is connected ever element $g\in G$ can be written as a finite product on elements of the form e^{x_i}, where $x_i\in\mathfrak{g}$. □

The assumption in proposition 3.3.2 may be too restrictive. For example, if the G-action Φ has a dense orbit in $\mathcal{M}$, the only G-invariant functions are constants. The following is a local version of 3.3.2 and follows by direct computation.

Proposition 3.3.4. *Let Φ be a G-symplectic action. Let $\mathcal{U}\subset\mathcal{M}$ be an open set in $\mathcal{M}$. If $f,g\in C^\infty(\mathcal{U})$ are locally invariant, namely:*

$$\mathcal{L}_{v_x} f = 0\,, \qquad \mathcal{L}_{v_x} h = 0\,, \qquad \forall\, x\in\mathfrak{g}\,. \tag{3.39}$$

Then $\{f,g\}$ is also locally invariant, i.e., $\mathcal{L}_{v_x}\{f,g\} = 0$.

Definition 3.3.5. The action Φ is said to be *weakly Hamiltonian* if it is symplectic and in addition $v_x\in\mathfrak{X}_{\mathrm{Ham}}(\mathcal{M})$ for every $x\in\mathfrak{g}$.

Let Φ be a symplectic action. Then Φ is weakly Hamiltonian if and only if there exists a basis $\{x_i\}$ of $\mathfrak{g}$ such that the corresponding fundamental vector fields v_{x_i} are Hamiltonian. We have then:

$$\iota_{v_{x_i}}\omega = -df_{v_{x_i}}\,, \tag{3.40}$$

for a unique (up to a constant) function $f_{v_{x_i}}\in C^\infty(\mathcal{M})$. For simplicity we will usually denote $f_{v_{x_i}}$ by f_{x_i}. The Hamiltonian function $\mu^*(x)$ of any element $x = \lambda^i x_i\in\mathfrak{g}$ is now given by:

$$\mu^*(x) = \lambda^i f_{x_i}\,. \tag{3.41}$$

[2]For simplicity we will sometimes omit the subscript specifying the point.

We obtain then a linear map $\mu^*\colon \mathfrak{g} \to C^\infty(\mathcal{M})$ defined by sending every $x \in \mathfrak{g}$ to its Hamiltonian function $\mu^*(x) := \mu^x \in C^\infty(\mathcal{M})$, which is unique up to the addition of a constant. We will call $\mu^*\colon \mathfrak{g} \to C^\infty(\mathcal{M})$ the *weakly comoment map* associated to the weakly Hamiltonian action Φ.

Definition 3.3.6. A weakly Hamiltonian action Φ is said to be *Hamiltonian* if it admits a weakly comoment map that is a Lie-algebra morphism from $(\mathfrak{g}, [\cdot,\cdot])$ into the Poisson algebra $(C^\infty(\mathcal{M}), \{\cdot,\cdot\})$.

Remark 3.3.7. A weakly Hamiltonian action need not be Hamiltonian. Let Φ be a weakly Hamiltonian action, and let $\mu^*\colon \mathfrak{g} \to C^\infty(\mathcal{M})$ be a weakly Hamiltonian comoment map for Φ. Then, there exists a unique map:

$$\tau\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}\,, \tag{3.42}$$

satisfying:

$$\mu^*([x,y]) - \{\mu^*(x), \mu^*(y)\} = \tau(x,y)\,, \tag{3.43}$$

for every $x, y \in \mathfrak{g}$. The map τ satisfies in addition:

$$\tau([x,y],z) + \tau([y,z],x) + \tau([z,x],y) = 0\,. \tag{3.44}$$

Therefore τ is a Lie algebra cocycle. Changing μ^* modifies τ by a coboundary and hence every weakly Hamiltonian action defines a Lie algebra cohomology class $[\tau] \in H^2(\mathfrak{g}, \mathbb{R})$.

Definition 3.3.8. A weakly Hamiltonian action Φ is is said to be Hamiltonian if $[\tau] = 0$. In that case $\mu^*\colon \mathfrak{g} \to C^\infty(\mathcal{M})$ is called the comoment map associated to Φ.

Proposition 3.3.9. *Let Φ be a Hamiltonian action with comoment map $\mu^*\colon \mathfrak{g} \to C^\infty(\mathcal{M})$. We have:*

$$\mu^*(\mathrm{Ad}_g\, x) = \mu^*(x) \circ \Phi_g\,, \qquad g \in G\,, \qquad x \in \mathfrak{g}\,, \tag{3.45}$$

Therefore, a comoment map can be understood as a lift of the $\mathfrak{g}$-action $x \mapsto v_x$:

$$\begin{tikzcd}
& (C^\infty(\mathcal{M}), \{\cdot,\cdot\}) \arrow[d, "\pi"] \\
\mathfrak{g} \arrow[ur] \arrow[r, "v_{-}"] & (\mathfrak{X}_{\mathrm{Ham}}(\mathcal{M}), [\cdot,\cdot])
\end{tikzcd}$$

where π denotes the canonical map sending each function to its unique hamiltonian vector field. In section 6.6.2 we will generalized the comoment map construction to the case of Lie groups acting on multisymplectic manifolds by multisymplectomorphisms.

Definition 3.3.10. Let $(\mathcal{M}, \omega)$ be a symplectic manifold and G a connected Lie group with Lie algebra $\mathfrak{g}$. Let us denote by $\mathfrak{g}^*$ the dual vector space of $\mathfrak{g}$ and assume that:

$$\Phi\colon G \times \mathcal{M} \to \mathcal{M}\,, \tag{3.46}$$

is a symplectc G-action on $(\mathcal{M}, \omega)$. Given $x \in \mathfrak{g}$, the action Φ induces a $\mathfrak{g}$-action $x \mapsto v_x$ by symplectic vector fields. The G-action Φ is said to be Hamiltonian if there exists a map:

$$\mu\colon \mathcal{M} \to \mathfrak{g}^*\,, \tag{3.47}$$

satisfying the following conditions:

1. For every $x \in \mathfrak{g}$ the corresponding vector field v_x satisfies:

$$d\mu^x = -\iota_{v_x}\omega\,, \tag{3.48}$$

   Here $\mu^x : \mathcal{M} \to \mathbb{R}$ is the function $\mu^x := \mu(x)$ on $\mathcal{M}$ given by the natural pairing of $\mathfrak{g}$ and $\mathfrak{g}^*$. Therefore, v_x must be Hamiltonian with Hamiltonian function given by μ^x.

2. $\mu\colon \mathcal{M} \to \mathfrak{g}^*$ is equivariant with respect to the given action Φ and the coadjoint action Ad^* of G on $\mathfrak{g}^*$, namely:

$$\mu \circ \Phi_g = \mathrm{Ad}^*_g \circ \mu\,, \qquad g \in G\,. \tag{3.49}$$

$(\mathcal{M}, \omega, G, \mu)$ is then called a Hamiltonian G-space with moment map μ. We have defined, for G connected, Hamiltonian actions as actions admitting a moment maps because if they are Hamiltonian and G is connected then a moment map is equivalent with a comoment map. Therefore the definition is, for G connected, equivalent to definition 6.1.13. If G is not connected, then the existence of moment map implies the existence of a comoment map but the existence of a comoment map only implies the existence of a moment map for the connected component of the identity in G.

Example 3.3.11. As an example let us consider let us consider the infinite-dimensional space of connections on the trivial principal bundle $P = \Sigma \times G$, where Σ is a compact Riemann surface and G is a compact Lie group [14]. The space $\mathcal{A} = \Omega^1(\Sigma, \mathfrak{g})$ of connections on P can be identified with one-forms on Σ taking values on $\mathfrak{g}$[3]. It can be seen that $\mathcal{A}$ is an infinite-dimensional Kähler manifold, with symplectic form and Kähler structure given by:

$$\omega|_A(\alpha, \beta) = \int_\Sigma < \alpha \wedge \beta >\,, \qquad \alpha \mapsto *\alpha\,, \qquad \alpha, \beta \in T_A\mathcal{A} \simeq \Omega^1(\Sigma, \mathfrak{g})\,. \tag{3.50}$$

where $A \in \mathcal{A}$ and $< \cdot, \cdot >$ is an invariant inner product on $\mathfrak{g}$. Since we assume P to be a trivial principal bundle, the gauge group is given by $\mathcal{G} = \mathrm{Map}(\Sigma, \mathcal{G})$ and it acts as usual on connections:

$$u \cdot A = u^{-1}Au + u^{-1}du\,. \tag{3.51}$$

Hence, we have an action $\Phi\colon \mathcal{G} \times \mathcal{A} \to \mathcal{A}$ which in fact preserves the Kähler structure on $\mathcal{A}$ and is Hamiltonian. For every $x \in \mathrm{Lie}(\mathcal{G}) \simeq \Omega^0(\Sigma, \mathfrak{g})$, the infinitesimal action is given by:

$$\begin{aligned} &\mathrm{Lie}(\mathcal{G}) \times \mathcal{A} \to T\mathcal{A}\,,\\ &(x, A) \mapsto (A, d_A x)\,, \end{aligned} \tag{3.52}$$

where $d_A\colon \Omega^0(\Sigma, \mathfrak{g}) \to \Omega^1(\Sigma, \mathfrak{g})$ denotes the connection induced by A on the trivial adjoint bundle of algebras. At each $A \in \mathcal{A}$ the corresponding comoment map is given by:

$$\mu^*|_A(x) = \int_\Sigma < F_A, x >\,, \qquad x \in \mathrm{Lie}(\mathcal{G})\,, \tag{3.53}$$

[3]Strictly speaking, $\mathcal{A}$ is an infinite-dimensional affine space modelled on $\Omega^1(\Sigma, \mathfrak{g})$.

where $F_A \in \Omega^2(\Sigma, \mathfrak{g})$ denotes the curvature of $A \in \mathcal{A}$. The associated moment map is thus given by:

$$\begin{aligned} \mu\colon \mathcal{A} &\to \Omega^2(\Sigma, \mathfrak{g})\,, \\ A &\mapsto F_A\,, \end{aligned} \tag{3.54}$$

where we have used $\mathrm{Lie}(\mathcal{G})^* \simeq \Omega^2(\Sigma, \mathfrak{g})$.

3.3.1 Symplectic reduction

One of the most important applications of the existence of a moment map is the canonical construction of a symplectic submanifold of the given symplectic manifold, which in addition is of physical importance in the description of the phase space of mechanical systems as symplectic manifolds. This construction is usually named in the literature as *symplectic reduction* or *Marsden-Weinstein quotient* [13].

Let $(\mathcal{M}, \omega)$ be a symplectic manifold admitting a moment map $\mu\colon M \to \mathfrak{g}^*$ for the action Φ of a Lie group G. We define:

$$\mathcal{M}_0 = \{p \in \mathcal{M} \mid \mu(p) = 0\}\ . \tag{3.55}$$

If $0 \in \mathfrak{g}^*$ is a regular value, namely if $d\mu_p\colon T_p\mathcal{M}_0 \to \mathfrak{g}^*$ is surjective for every $p \in \mathcal{M}_0$, then $\mathcal{M}_0$ is a closed submanifold of $\mathcal{M}$. Using that he moment map is equivariant and that 0 is a fixed point of the coadjoint action of G we conclude that G preserves $\mathcal{M}_0$, and therefore $\Phi\colon G \times \mathcal{M} \to \mathcal{M}$ induces a G-action $\Phi_0\colon G \times \mathcal{M}_0 \to \mathcal{M}_0$ on $\mathcal{M}_0$. Therefore, we can define the orbit space by this action:

$$\mathcal{M}_0^G = \mathcal{M}_0/G\,, \tag{3.56}$$

which in general is not guaranteed to be a smooth manifold unless the action satisfies some particular conditions, for example being free and proper. We will consider only the case in which:

- The value $0 \in \mathfrak{g}$ is regular and hence $\mathcal{M}_0 \subset \mathcal{M}$ is a closed submanifold of $\mathcal{M}$.
- The group G acts freely and properly on $\mathcal{M}_0$, so $\mathcal{M}_0/G$ is again a manifold.

Under the two assumptions stated above we have the following result.

Proposition 3.3.12. *The submanifold $\mathcal{M}_0 \subset \mathcal{M}$ is coisotropic and the corresponding isotropic foliation is given by the orbits of the G-action $\Phi_0\colon G \times \mathcal{M}_0 \to \mathcal{M}_0$. The quotient:*

$$\mathcal{M}//G := \mathcal{M}_0/G\,, \tag{3.57}$$

is a symplectic manifold of dimension $\dim \mathcal{M}//G = \dim \mathcal{M} - 2\dim G$.

Definition 3.3.13. The quotient $\mathcal{M}//G$ is called the *Marsden-Weinstein quotient.*

Chapter 4

Homological algebra

In this chapter we will consider various aspects of homological algebra and category theory. The main goal of this chapter is to introduce some of the background that will be needed through the rest of the document as well as to present some results of intrinsic interest. Homological algebra theory studies homology (or co-homology) in an abstract setting, that is, on abstractly defined complexes (or co-complexes), and it is intimately related to category theory, which provides the natural language where to pose and solve many of the relevant problems.

4.1 Graded algebras and coalgebras

The purpose of this section is to introduce the elements of graded algebra theory that will be needed through the rest of the letter. Graded algebra theory simply refers to the study of algebraic structures in a graded vector space.

4.1.1 Graded algebras

We begin with a series of definitions increasing step by step the structures involved. We begin by the simple definition of an algebra.

Definition 4.1.1. A *real algebra* $(V, \cdot)$ is a real vector space V space equipped with a bilinear product $V \times V \to V$, denoted by $\cdot$ or concatenation of elements.

We require the product $\cdot$ of an algebra to be inner. It may have, however, other properties. For example, if $x_1 \cdot x_2 = x_2 \cdot x_1$ for all $x_1, x_2 \in V$ the algebra $(V, \cdot)$ is said to be commutative. If $x_1 \cdot (x_2 \cdot x_3) = (x_1 \cdot x_2) \cdot x_3$ for all $x_1, x_2, x_3 \in \mathrm{X}$ then the algebra is said to be associative. If the algebra V contains an identity, that is, an element $e \in V$ such that $e \cdot x = x \cdot e = x$ for all $x \in V$ then it is said to be unital. A real algebra such that the product is associative and has an identity is therefore a ring that is also a vector space, and it is called a *unital associative algebra*.

Definition 4.1.2. Let G be an abelian group. A *G-graded vector space V* over $\mathbb{R}$ is a collection $(V_g)_{g \in G}$ of vector spaces over $\mathbb{R}$.

The homogeneous elements of degree $g \in G$ of a G-graded vector space V are the elements of V_g. In this work we will consider exclusively $G = \mathbb{Z}$-graded vector spaces $V = \bigoplus_{i \in \mathbb{Z}} V_i$. The grade of an homogeneous element $x \in V$ is denoted by $|x|$, and of course, since we will consider only $\mathbb{Z}$-graded vector spaces, we will always have $|x| \in \mathbb{Z}$.

Definition 4.1.3. A real graded algebra is a real graded vector space $V = \bigoplus_{i \in \mathbb{Z}} V_i$ equipped with a bilinear product $V \times V \to V$ which will be denoted by $\cdot$ or concatenation of elements, such that:

$$V_i \cdot V_j \subset V_{i+j}\,. \tag{4.1}$$

Definition 4.1.4. A real graded Lie algebra is a real graded algebra $(V, \cdot)$ equipped with a bilinear product $[\cdot,\cdot] : V \times V \to V$ such that the following axioms are satisfied:

1. $[\cdot,\cdot]$ respects the grading of V, that is, $[V_i, V_j] \subseteq V_{i+j}$.
2. If $x_1, x_2 \in V$ are homogeneous elements then
$$[x_1, x_2] = -(-1)^{|x_1||x_2|}\,[x_2, x_1]\,. \tag{4.2}$$
That is, $[\cdot,\cdot]$ is antisymmetric in the graded sense, i.e., it is graded antisymmetric.
3. If $x_1, x_2, x_3 \in V$ then:
$$(-1)^{|x_1||x_3|}[x_1,[x_2,x_3]] + (-1)^{|x_1||x_2|}[x_2,[x_3,x_1]] + (-1)^{|x_2||x_2|}[x_3,[x_1,x_2]] = 0\,. \tag{4.3}$$
That is, $[\cdot,\cdot]$ satisfies the graded Jacobi identity.

Equation (4.2) is just the graded version of the antisymmetric Lie bracket of a not-graded Lie algebra. Analogously, Eq. (4.3) is just the graded version of the Jacobi identity for not-graded Lie algebras. When V is concentrated in degree zero, a real graded Lie algebra is just an ordinary real Lie algebra. Let us consider now a simple example taken from physics.

Example 4.1.5. **Supersymmetry algebra**. The Super-Poincaré algebra $\mathfrak{sp}$ is, in physical terms, an extension of the Poincaré algebra by fermionic generators that obey anti-commutation relations. We denote by M_{ab} the generators of the Lorentz group, by P_a the generators of translations and by Q_α the fermionic generators, which are Spin$(1,3)$ spinors of definite chirality. They obey the following (anti)-commutation relations:

$$\begin{aligned}
[M_{ab}, M_{cd}] &= -M_{eb}\Gamma_{\mathrm{v}}\,(M_{cd})_a^{\;e} - M_{ae}\Gamma_{\mathrm{v}}\,(M_{cd})_b^{\;e} && (4.4)\\
[P_a, M_{cd}] &= -P_e\Gamma_{\mathrm{v}}\,(M_{cd})_a^{\;e} && (4.5)\\
[Q^\alpha, M_{ab}] &= \Gamma_{\mathrm{s}}\,(M_{ab})_\beta^{\;\alpha}\,Q^\beta && (4.6)\\
\left\{Q^\alpha, Q^\beta\right\} &= i\left(\gamma^a C^{-1}\right)^{\alpha\beta} P_a\,, && (4.7)
\end{aligned}$$

where Γ_{v} denotes the vectorial representation, Γ_{s} denotes the spinorial representation and C is the charge conjugation matrix. The *anticommutator* $\{\cdot,\cdot\}$ is defined as $\left\{Q^\alpha, Q^\beta\right\} = Q^\alpha Q^\beta + Q^\beta Q^\alpha$. The Super-Poincaré algebra gets beautifully described as a particular instance of $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$[1] graded Lie algebra $\mathfrak{sp} = \mathfrak{sp}_0 \oplus \mathfrak{sp}_1$ with bracket $[\cdot,\cdot]_{\mathbb{Z}_2}$. The elements of $\mathfrak{sp}_0$ are called even and correspond to the *bosonic* generators M_{ab} and P_a of the algebra. The elements of $\mathfrak{sp}_1$ are called odd and correspond to the *fermionic* generators of the algebra Q_α. In particular, we have:

$$\begin{aligned}
[x_1, x_2]_{\mathbb{Z}_2} &= -[x_2, x_1]_{\mathbb{Z}_2}\,, && \forall\, x_1, x_2 \in \mathfrak{sp}_0\,, && (4.8)\\
[x_1, x_2]_{\mathbb{Z}_2} &= -[x_2, x_1]_{\mathbb{Z}_2}\,, && \forall\, x_1 \in \mathfrak{sp}_0\,,\ \forall\, x_2 \in \mathfrak{sp}_1\,, && (4.9)\\
[x_1, x_2]_{\mathbb{Z}_2} &= [x_2, x_1]_{\mathbb{Z}_2}\,, && \forall\, x_1, x_2 \in \mathfrak{sp}_1\,. && (4.10)
\end{aligned}$$

Therefore, equation (4.8) corresponds to the commutators (4.4) and (4.5) in the Super-Poincaré algebra, equation (4.9) corresponds to the commutator (4.6) in the Super-Poincaré algebra, and equation (4.10) corresponds to the anti-commutator (4.7) in the Super Poincaré algebra.

[1]The definition of $\mathbb{Z}_2$-graded space can be obtained from the definition of $\mathbb{Z}$-graded vector space by simply substituting $\mathbb{Z}$ by $\mathbb{Z}_2$.

Definition 4.1.6. Let $(V, \cdot)$ be a graded algebra. A homogeneous linear map $d : V \to V$ of grade $|d|$ on V is called a homogeneous derivation if $d(x_1 \cdot x_2) = d(x_1)x_2 + \epsilon^{|x_1||d|}x_1 \cdot d(x_2)$, where $\epsilon = \pm 1$ and $x_1, x_2 \in V$ are homogeneous elements. A graded derivation is sum of homogeneous derivations with the same ϵ. In the context of graded algebra, the choice $\epsilon = -1$ is the most natural one, since it takes into account the graded structure of the algebra.

Definition 4.1.7. A real differential graded Lie algebra is a real graded Lie algebra $(V, \cdot)$ equipped with a degree ± 1 (depending on chain or cochain complex convention) derivation $d : V \to V$ that satisfies:

1. $d \circ d = 0$. Therefore d gives V the structure of a chain ($|d| = -1$) or cochain complex ($|d| = 1$).
2. $d[x_1, x_2] = [dx_1, x_2] + (-1)^{|x_1|}[x_1, dx_2]$, where x_1 and x_2 are homogeneous elements of V.

Given two homogeneous elements $x_1, x_2 \in V$ of an arbitrary graded algebra $(V, \cdot)$, in principle $x_1 \cdot x_2$ and $x_2 \cdot x_1$ are different elements of $(V, \cdot)$ not related in any particular way. However, if for every pair of homogeneous elements $x_1, x_2 \in V$ the following holds:

$$x_1 \cdot x_2 = (-1)^{|x_1||x_2|}x_2 \cdot x_1 \,, \tag{4.11}$$

then $(V, \cdot)$ is said to be a graded commutative algebra. Analogously, if

$$x_1 \cdot x_2 = -(-1)^{|x_1||x_2|}x_2 \cdot x_1 \,, \tag{4.12}$$

holds, then the algebra $(V, \cdot)$ is said to be a graded anti-commutative algebra. This procedure can be generalized by defining the *Koszul sign.* Let $x_1, \dots, x_n$ be elements of a symmetric graded algebra $(V, \cdot)$ and $\sigma \in \Sigma_n$ a permutation. The Koszul sign $\epsilon(\sigma) = \epsilon(\sigma; x_1, \dots, x_n)$ is defined by the equality:

$$x_1 \cdots x_n = \epsilon(\sigma; x_1, \dots, x_n)x_{\sigma(1)} \cdots x_{\sigma(n)} \,,$$

which holds in the free graded commutative algebra generated by V, with product denoted by concatenation of elements. The Koszul sign can be equivalently defined using an antisymmetric graded algebra $(V, \cdot)$ as follows:

$$x_1 \cdots x_n = (-1)^{\sigma}\epsilon(\sigma; x_1, \dots, x_n)x_{\sigma(1)} \cdots x_{\sigma(n)} \,,$$

Given $\sigma \in \Sigma_n$, $(-1)^{\sigma}$ denotes the usual sign of a permutation. Notice that $\epsilon(\sigma)$ does not include the sign $(-1)^{\sigma}$. For example, given $x_1, x_2, x_2 \in V$, where $(V, \cdot)$ is free graded commutative algebra, we have:

$$x_1 \cdot x_2 \cdot x_3 = \epsilon(3, 2, 1; x_1, x_2.x_3)x_3 \cdot x_1 \cdot x_2 \,, \qquad \epsilon(3, 2, 1; x_1, x_2.x_3) = (-1)^{|x_3||x_2|+|x_3||x_1|} \,. \tag{4.13}$$

We say that $\sigma \in \Sigma_{p+q}$ is a $(\mathbf{p}, \mathbf{q})$-*unshuffle* if and only if σ is a permutation of a set of $(p + q)$ elements such that $\sigma(1) < \cdots < \sigma(p)$ and $\sigma(p + 1) < \cdots < \sigma(p + q)$. The set of (p, q)-unshuffles is denoted by $\mathrm{Sh}(p, q)$. For example, $\mathrm{Sh}(2, 1)$ is the set of cycles $\{(1), (23), (123)\}$.

If V is a graded vector space, then $\mathrm{s}V$ denotes the suspension of V, and $\mathrm{s}^{-1}V$ denotes the desuspension of V, defined respectively by:

$$(\mathrm{s}V)_i = V_{i-1} \,, \qquad \left(\mathrm{s}^{-1}V\right)_i = V_{i+1} \,. \tag{4.14}$$

Another very used notation for the (de)suspension of a graded vector space V is:

$$\mathrm{s}^{\pm k}V = V[\mp k] \,. \tag{4.15}$$

Sometimes we will write $s^{\pm 1}x$ where x is an homogeneous element of a given graded vector space V. The meaning of such expression can be understood as follows. Let us assume that $x \in V_i$, $i \in \mathbb{Z}$. Then we have:

$$s^{\pm 1}x \in s^{\pm 1}(V_i) = \left(s^{\pm 1}V\right)_{i\mp 1}, \tag{4.16}$$

since $\left(s^{\pm 1}V\right)_{i\mp 1} = V_i$.

Definition 4.1.8. A morphism f from a $\mathbb{Z}$-graded vector space V to a $\mathbb{Z}$-graded vector space W is a collection of linear maps $(f_i : V_i \to W_i)_{i\in\mathbb{Z}}$.

Hence, it is implicitly assumed in the definition that a morphism preserves the grading, that is, that $|f(x)| = |x| \in \mathbb{Z}$. It is said then that f is homogeneous of degree zero. An isomorphism of graded vector spaces is a morphism f whose components f_i are isomorphisms of vector spaces. It is possible, of course, to consider maps that do not preserve the grading. The degree of a map f is denoted by $|f|$. If a map $f : V \to W$ of graded vector spaces has degree $|f| = k$, $k \in \mathbb{Z}$, then we have

$$f(V_i) \subseteq V_{i+k}, \ \forall\, i \in \mathbb{Z}. \tag{4.17}$$

The set of all morphisms from a graded vector space V to a graded vector space W is denoted by $\mathrm{Hom}\,(V, W)$. Notice that $\mathrm{Hom}\,(V, W)$ is itself a graded vector space with homogeneous component $\mathrm{Hom}\,(V, W)_i$ $i \in \mathbb{Z}$ given by the set of components $f_i\colon V_i \to W_i$, $f \in \mathrm{Hom}\,(V, W)$.

Definition 4.1.9. Let $F(V\times W)$ the free vector space over $\mathbb{R}$ whose generators are the points of $V \times W$, where $\times$ stands for the Cartesian product [2] The tensor product is defined as a certain quotient vector space of $F(V \times W)$. Consider the subspace R of $F(V \times W)$ generated by the following elements[3]

$$\begin{aligned}
(v_1, w) + (v_2, w) - (v_1 + v_2, w)\,, \\
(v, w_1) + (v, w_2) - (v, w_1 + w_2)\,, \\
c \cdot (v, w) - (cv, w)\,, \\
c \cdot (v, w) - (v, cw)\,,
\end{aligned} \tag{4.18}$$

where $v, v_1, v_2 \in V$, $w, w_1, w_2 \in W$, and $c \in \mathbb{R}$. The tensor product is then defined as the vector space

$$V \otimes W \equiv F(V \times W)/R\,. \tag{4.19}$$

The tensor product of two vectors v and w is denoted by the equivalence class $v \otimes w \in ((v, w) + R)$, where $v \otimes w \in V \otimes W$. The principal effect of taking the quotient by R in the free vector space is that the following equations hold in $V \otimes W$

$$\begin{aligned}
(v_1 + v_2) \otimes w &= v_1 \otimes w + v_2 \otimes w\,, \\
v \otimes (w_1 + w_2) &= v \otimes w_1 + v \otimes w_2\,, \\
cv \otimes w &= v \otimes cw = c(v \otimes w)\,.
\end{aligned} \tag{4.20}$$

The tensor product of two graded vector spaces V and W is defined to be another graded vector space $V \otimes W$ with grading

[2] The Cartesian product $V \times W$ is the set of pairs (v, w), $v \in V$, $w \in W$, which is itself a vector space, although here it is considered merely as a set.

[3] To simplify the notation, we denote by (v, w) the element $1 \cdot (u, w) \in F(V \times W)$.

$$(V \otimes W)_i = \bigoplus_{i=j+k} V_j \otimes W_k\,, \qquad i = j+k\,. \tag{4.21}$$

The tensor product can be also applied to morphisms $f, g \in \mathrm{Hom}\,(V, W)$, and it is given by

$$(f \otimes g)\,(v \otimes w) = (-1)^{|g||f|} f(v) \otimes g(w)\,. \tag{4.22}$$

Definition 4.1.10. Let V be a graded vector space. The *tensor algebra* $\mathcal{T}(V)$ is the graded vector space given by the collection of vector spaces

$$\mathcal{T}(V)_m = \bigoplus_{k \geq 0} \bigoplus_{j_1+\cdots+j_k=m} V_{j_1} \otimes \cdots \otimes V_{j_k}\,, \qquad m \in \mathbb{Z}\,. \tag{4.23}$$

For $k=0$ the corresponding summand is set to be equal to $\mathbb{R}$.
Every component $\mathcal{T}(V)_m$ can be decomposed with respect to the tensor product degree $\otimes$ as follows

$$\mathcal{T}^k(V)_m \equiv \mathcal{T}(V)_m \cap V^{\otimes k}\,, \qquad k \geq 1\,. \tag{4.24}$$

For $k=0$ we have $\mathcal{T}^0(V) = \mathbb{R}$. The degree of an element $x_1 \otimes \cdots \otimes x_k \in V_1 \otimes \cdots \otimes V_k$ is defined as $|x_1 \otimes \cdots \otimes x_k| = \sum_{i=1}^k |x_i|$.
The vector space $\mathcal{T}^k(V)_m$ carries two natural actions, *even* and *odd*, of the group Σ_k of permutations of a set of k elements. The *even* representation intuitively corresponds to the elements of $\mathcal{T}^k(V)_m$ symmetric in the graded sense. It is defined by

$$\sigma \cdot (x_1 \otimes \cdots \otimes x_k) \equiv (-1)^{|x_i||x_{(i+1)}|} x_1 \otimes \cdots \otimes x_{(i+1)} \otimes x_i \otimes \cdots \otimes x_k\,, \tag{4.25}$$

where σ is the transposition of the i'th and the $(i+1)$'th element. Similarly, the odd representation intuitively corresponds to the antisymmetric elements of $\mathcal{T}^k(V)_m$ in the graded sense. It is defined by

$$\sigma \cdot (x_1 \otimes \cdots \otimes x_k) \equiv -(-1)^{|x_i||x_{(i+1)}|} x_1 \otimes \cdots \otimes x_{(i+1)} \otimes x_i \otimes \cdots \otimes x_k\,, \tag{4.26}$$

where σ is the transposition of the i'th and the $(i+1)$'th element. Notice that the even as well as the odd representations can be defined for an arbitrary element $\sigma \in \Sigma_k$ as follows

$$\sigma \cdot (x_1 \otimes \cdots \otimes x_k) \equiv \epsilon\,(\sigma; x_1, \cdots, x_k)\, x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(k)}\,, \tag{4.27}$$

for the even representation and

$$\sigma \cdot (x_1 \otimes \cdots \otimes x_k) \equiv (-1)^{\sigma} \epsilon\,(\sigma; x_1, \cdots, x_k)\, x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(k)}\,, \tag{4.28}$$

for the odd representation.

Definition 4.1.11. Given a graded vector space V, the *graded symmetric algebra* $\mathcal{S}(V)$ of V is the graded vector space whose elements are the invariants or the even representation of Σ on $\mathcal{T}(V)$ with the inherited grading.

Definition 4.1.12. Given a graded vector space V, the *graded antisymmetric algebra* $\Lambda(V)$ of V is the graded vector space whose elements are the invariants or the odd representation of Σ on $\mathcal{T}(V)$ with the inherited grading.
We will denote by $\mathcal{S}^k(V)$ and $\Lambda^k(V)$ the homogeneous elements of degree k of $\mathcal{S}(V)$ and $\Lambda(V)$ respectively.

We can extend the action of $\mathrm{s}^{\pm 1}$ to the tensor product of an arbitrary number of graded vector spaces $V_1, \ldots, V_k$ as follows

$$\begin{aligned} \mathrm{s}^{\pm k}: \quad & V_1 \otimes \cdots \otimes V_k \to \mathrm{s}^{\pm 1} V_1 \otimes \cdots \otimes \mathrm{s}^{\pm 1} V_k \\ & x_1 \otimes \cdots \otimes x_k \mapsto (-1)^{\sum_{i=1}^{k}(k-i)|x_i|} \mathrm{s}^{\pm 1} x_1 \otimes \cdots \otimes \mathrm{s}^{\pm 1} x_k \,. \end{aligned} \tag{4.29}$$

Notice that the sign $(-1)^{\sum_{i=1}^{k}(k-i)|x_i|}$ can be understood by considering $s^{\pm 1}$ as an odd element which requires the introduction of the sign $(-1)^{|x_i|}$ every time it jumps over x_i when acting on $x_1 \otimes \cdots \otimes x_k$. $s^{\pm k}$ is an isomorphism of vector spaces which is not an isomorphism of graded vector spaces, since it does not preserve the grading. Defining now $\mathrm{dec}_k : V^{\otimes^k} \to V^{\otimes^k}$ by

$$\mathrm{dec}_k \left(x_1 \otimes \cdots \otimes x_k \right) \equiv (-1)^{\sum_{i=1}^{k}(k-i)|x_i|} x_1 \otimes \cdots \otimes x_k \,, \tag{4.30}$$

we obtain the so-called *décalage-isomorphism* between $\mathrm{s}^{\pm k} \mathcal{S}^k (V)$ and $\Lambda^k \left(\mathrm{s}^{\pm 1} V\right)$. The décalage-isomorphism preserves the grading and therefore $\mathrm{s}^{\pm k} \mathcal{S}^k (V)$ and $\Lambda^k \left(\mathrm{s}^{\pm 1} V\right)$ are isomorphic not only as vector spaces but also as graded vector spaces.

Since it will be useful later, we will rewrite now the Koszul sign $\epsilon\left(\sigma; s^{-1}x_1, \ldots, s^{-1}x_n\right)$ in terms of $\epsilon\left(\sigma; x_1, \ldots, x_n\right)$. In order to do so, we notice that if $(x_1, \ldots, x_k) \in \Lambda^k V$ then

$$(x_1, \ldots, x_k) = (-1)^{\sigma} \epsilon\left(\sigma; x_1, \ldots, x_k\right) \left(x_{\sigma(1)}, \ldots, x_{\sigma(k)}\right) \,, \tag{4.31}$$

and therefore

$$s^{-k}\left((x_1, \ldots, x_k)\right) = (-1)^{\sigma} \epsilon\left(\sigma; x_1, \ldots, x_k\right) s^{-k} \left(\left(x_{\sigma(1)}, \ldots, x_{\sigma(k)}\right)\right) \,, \tag{4.32}$$

which implies, using equation (4.29)

$$\begin{aligned} (-1)^{\sum_{i=1}^{k}(k-i)|x_{\sigma(i)}|} (-1)^{\sigma} \epsilon\left(\sigma; x_1, \ldots, x_k\right) \epsilon\left(\sigma; s^{-1}x_1, \ldots, s^{-1}x_k\right) \left(s^{-1}x_1, \ldots, s^{-1}x_k\right) & \\ = (-1)^{\sum_{i=1}^{k}(k-i)|x_{\sigma(i)}|} \left(s^{-1}x_1, \ldots, s^{-1}x_k\right) \,. & \end{aligned} \tag{4.33}$$

Hence we conclude

$$\epsilon\left(\sigma; s^{-1}x_1, \ldots, s^{-1}x_k\right) = (-1)^{\sum_{i=1}^{k}(k-i)\left(|x_i| + |x_{\sigma(i)}|\right)} (-1)^{\sigma} \epsilon\left(\sigma; x_1, \ldots, x_k\right) \,. \tag{4.34}$$

We finish this section by defining two specific kind of algebras that we will find later. When we introduce the *Cartan calculus* in section 2.1.1, we will encounter a particular instance of *Gerstenhaber algebra*, which is defined as follows

Definition 4.1.13. A Gerstenhaber algebra is an associative and commutative graded algebra $(V, \cdot)$ equipped with a degree -1 bilinear map $[\cdot, \cdot] : V \to V$ such that the following conditions hold

1. $[x_1, x_2] = -(1)^{(|x_1|-1)(|x_2|-1)} [x_2, x_1]$, for any two homogeneous elements $x_1, x_2 \in V$. That is, the bilinear map is antisymmetric in the graded sense in $s^{-1}V$.

2. $[x, \cdot]$ is a derivation on V of degree $|x| - 1$ for every $x \in V$.

3. The bilinear map obeys

$$[x_1, [x_2, x_3]] = [[x_1, x_2], x_3] + (1)^{(|x_1|-1)(|x_2|-1)} [x_2, [x_3, x_1]] \,, \tag{4.35}$$

   which becomes the graded Jacobi identity on $s^{-1}V$.

From the definition In the context of symplectic geometry in section 3.2 we will find a particular example of a Poisson algebra, whose definition is given by

Definition 4.1.14. A Lie algebra $(X, \cdot, [\cdot,\cdot])$ is called a Poisson algebra if X has a commutative, associative algebra structure such that

$$[x_1 x_2, x_3] = x_1 [x_2, x_3] + [x_1, x_3] x_2 \,, \tag{4.36}$$

for all $x_1, x_2, x_3 \in X$.

4.1.2 Graded coalgebras

Coalgebras are structures that are dual, in the category-theoretic sense of reversing arrows, to algebras. An algebra $(V, \cdot)$ is, as explained in section 4.1.1, a vector space equipped with a product $\cdot$, which defines the following application

$$\begin{aligned} \cdot : V \otimes V &\to V \\ (x_1, x_2) &\mapsto x_1 \cdot x_2 \,. \end{aligned} \tag{4.37}$$

Therefore, we should expect a coalgebra C to be equipped with some map in the opposite direction

$$C \to C \otimes C \,. \tag{4.38}$$

The precise definition goes as follows.

Definition 4.1.15. A graded coalgebra (C, Δ) is a graded vector space C equipped with a linear map $\Delta \colon C \to C \otimes C$, the so-called *comultiplication*, such that

$$\Delta (C_i) \subset \bigoplus_{j+k=i} C_j \otimes C_k \,. \tag{4.39}$$

A coalgebra (C, Δ) is said to be *coassociative* if the following condition holds

$$(\Delta \otimes \mathrm{id})\, \Delta = (\mathrm{id} \otimes \Delta)\, \Delta \,. \tag{4.40}$$

Equation (4.40) is just the coalgebra version of the associativity condition for an algebra. Similarly, we can define a *cocommutative* coalgebra by imposing the dual condition of commutativity in an algebra. Indeed, if (C, Δ) is a coalgebra, let us denote by $T : C \otimes C \to C$ the twist map $T\,(x \otimes y) = (-1)^{|x||y|} y \otimes x$. It is said that (C, Δ) is cocommutative if and only if $T \circ \Delta = \Delta$. As well as a the concept of indentity can be defined for algebras, we can introduce a similar structure for coalgebras, called the *identity*.

Definition 4.1.16. A counit for (C, Δ) is a linear map $\epsilon : C \to \mathbb{R}$ such that $(\epsilon \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \epsilon)\Delta = \mathrm{id}$.

In order to elucidate the structure present in a coalgebra, the following commutative diagrams may be illustrative

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "\mathrm{id} \otimes \Delta"] \\ C \otimes C \arrow[r, "\Delta \otimes \mathrm{id}"] & C \otimes C \otimes C \end{tikzcd}$$

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] \arrow[dr, "\mathrm{id}"] & C \otimes C \arrow[d, "\mathrm{id} \otimes \epsilon"] \\ C \otimes C \arrow[r, "\epsilon \otimes \mathrm{id}"] & \mathbb{R} \otimes C \simeq C \simeq C \otimes \mathbb{R} \end{tikzcd}$$

Diagram (4.1.16) is the dual, in the category-theoretic sense of reversing arrows, of the analogous diagram that express associativity of algebra multiplication. Diagram (4.1.16) is the dual, in the same sense as before, of the analogous diagram expressing the existence of an identity in a unital algebra.

Definition 4.1.17. A coalgebra (C, Δ) with counit ϵ is *coaugmented* if and only if it can be equipped with an injective linear map

$$\begin{aligned} \mathbb{R} &\hookrightarrow C && (4.41) \\ 1 &\mapsto 1_C && (4.42) \end{aligned}$$

such that $\epsilon(1_C) = 1$ and $\Delta(1_C) = 1_C \otimes 1_C$. We can write in that case $\bar{C} = \ker \epsilon$ so that we have $C \simeq \mathbb{R} \oplus \bar{C}$ as vector spaces.

Intuitively, if a coalgebra C admits a coaugmentation, then it contains a copy of $\mathbb{R}$. For a given coaugmented coalgebra, we define the *reduced comultiplication* $\bar{\Delta} \colon \bar{C} \to \bar{C} \otimes \bar{C}$ as follows

$$\bar{\Delta} c = \Delta c - c \otimes 1_C - 1_C \otimes c. \tag{4.43}$$

The equation above makes $\bar{C}$ into a coalgebra with no counit. We denote now the **reduced diagonal** by $\Delta^{(n)}$. It can be recursively defined as follows

$$\begin{aligned} \bar{\Delta}^{(0)} &= \mathrm{id} \\ \bar{\Delta}^{(1)} &= \bar{\Delta} \\ \bar{\Delta}^{(n)} &= \left(\bar{\Delta} \otimes \mathrm{id}^{\otimes(n-1)}\right) \circ \bar{\Delta}^{(n-1)} \colon \bar{C} \to \bar{C}^{\otimes(n+1)}. \end{aligned} \tag{4.44}$$

It can be shown by induction that we can rewrite $\bar{\Delta}^{(n)}$ as

$$\bar{\Delta}^{n} = \left(\bar{\Delta}^{(n-1)} \otimes \mathrm{id}\right) \circ \bar{\Delta}. \tag{4.45}$$

A coaugmented coalgebra $(C, \Delta, \epsilon, 1_C)$ has always a canonical filtration[4] which can be defined recursively as follows

$$\begin{aligned} F_0 C &= \mathbb{R} \cdot 1_C & (4.46)\\ F_k C &= \{x \in \bar{C} \mid \bar{\Delta} x \in F_{k-1} C \otimes F_{k-1} C\}\ , & (4.47) \end{aligned}$$

and it is *connected* if and only if

$$C = \bigcup F_k C\,. \tag{4.48}$$

If $(C, \Delta, \epsilon, 1_C)$ is connected, it can be proven that the coaugmentation 1_C is unique [5].

Example 4.1.18. Graded symmetric algebra. Let V be a graded vector space. Then the the graded symmetric algebra is given by

$$S(V) = \mathbb{R} \oplus \sum_{k=1}^{\infty} S^k V = \mathbb{R} \oplus \bar{S}(V) \tag{4.49}$$

is a coaugmented cocommutative coalgebra in a natural way. The comultiplication Δ is defined as the unique morphism of algebras such that $\Delta(v) = v \otimes 1 + 1 \otimes v$ holds for all $v \in V$, assuming that $\Delta(1) = 1 \otimes 1$. The counit is defined as the projection $S(V) \to \mathbb{R}$, and the coaugmentation is given by the inclusion $\mathbb{R} \hookrightarrow S(V)$. The reduced comultiplication $\bar{\Delta}$ on $\bar{S}(V)$ can be given explicitly by

$$\begin{aligned} \bar{\Delta}(v_1 \odot v_2 \odot \cdots \odot v_n) &= \sum_{1 \le p \le n-1} \sum_{\sigma \in \mathrm{Sh}(p, n-p)} \epsilon(\sigma) \left(v_{\sigma(1)} \odot v_{\sigma(2)} \odot \cdots \odot v_{\sigma(p)}\right) \\ &\otimes \left(v_{\sigma(p+1)} \odot v_{\sigma(p+2)} \odot \cdots \odot v_{\sigma(n)}\right)\ , \end{aligned} \tag{4.50}$$

where $\odot$ denotes the symmetrized tensor product. That is, if $v_1, \cdots, v_p \in V$ are $p \in \mathbb{N}$ homogeneous elements then we have

$$v_1 \odot \cdots \odot v_p = \frac{1}{p!} \sum_{\sigma \in \Sigma_p} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(p)}\,. \tag{4.51}$$

The following lemma, which we extract from [69], will be needed in order to properly rewrite L_∞-morphisms.

Lemma 4.1.19. *If $v_1 \odot v_2 \odot \cdots \odot v_n \in \bar{S}(V)$, and $1 \le p \le n-1$ then*

$$\begin{aligned} \bar{\Delta}^p(v_1 \odot \cdots \odot v_n) &= \sum_{k_1, k_2, \ldots, k_{p+1} \ge 1}^{k_1 + k_2 + \cdots + k_{p+1} = n} \sum_{\sigma \in \mathrm{Sh}(k_1, k_2, \ldots, k_{p+1})} \epsilon(\sigma)\, v_{\sigma(1)} \odot \cdots \odot v_{\sigma(k_1)} \\ &\otimes v_{\sigma(k_1+1)} \odot \cdots \odot v_{\sigma(k_1+k_2)} \otimes v_{\sigma(k_1+k_2+1)} \odot \cdots \odot v_{\sigma(k_1+k_2+k_3)} \otimes \cdots \\ &\otimes v_{\sigma(m-k_{p+1}+1)} \odot \cdots \odot v_{\sigma(n)}\ , \end{aligned} \tag{4.52}$$

and in particular we have

$$\bar{\Delta}^{(n-1)}(v_1 \odot \cdots \odot v_n) = \sum_{\sigma \in S_n} \epsilon(\sigma) v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)}\,. \tag{4.53}$$

[4] A filtration is an indexed set S_i of sub-objects of a given algebraic structure S, with the index i taking values in a totally ordered set I, subject to the condition that if $i \le j$ in I then $S_i \subseteq S_j$.

[5] See proposition 3.1 in section B3 of reference [68]

Proof. See lemma A.1 in [69].
Lemma 4.1.19 implies that $\ker \bar{\Delta}^{(k)} = \bar{S}^{\bullet \leq k}(V)$ for $k \geq 0$ and also that

$$\bar{S}(V) = \bigcup_n \ker \bar{\Delta}^{(n)} . \tag{4.54}$$

The filtration $F_n S(V)$ corresponds, for $n \geq 1$, to the filtration on $\bar{S}(V)$ defined by $\ker \bar{\Delta}^{(n)}$. This proves that $S(V)$ is a connected coalgebra.

In order to define L_∞-algebras and L_∞-algebra morphisms, the concepts of coalgebra differential and coalgebra morphism are going to prove essential.

Definition 4.1.20. A *codifferential* of degree one on a coalgebra (C, Δ) is a linear map $Q\colon C^i \to C^{i+1}$ satisfying

$$Q \circ Q = 0 , \tag{4.55}$$

and the coLeibniz identity

$$\Delta Q = (Q \otimes \mathrm{id})\,\Delta + (\mathrm{id} \otimes Q)\Delta . \tag{4.56}$$

In the case of a codifferential Q defined on connected coalgebra $(C, \Delta, \epsilon, 1_C)$, we require an additional condition, namely

$$Q(1_C) = 0 . \tag{4.57}$$

A codifferential defined on a connected coalgebra $(C, \Delta, \epsilon, 1_C)$ is uniquely given by its restriction to $\bar{C}$, which satisfies the coLeibniz identity with respect to $\bar{\Delta}$.

Let us consider the particular case $C = \bar{\mathcal{T}}(V)$. Given a codifferential Q on $\bar{\mathcal{T}}(V)$, consider the restrictions

$$Q_m = Q|_{\bar{\mathcal{T}}^m(V)} : \bar{\mathcal{T}}^m(V) \to \bar{\mathcal{T}}(V), \quad 1 \leq m < \infty , \tag{4.58}$$

so that $Q = \sum_k^\infty Q_k$, and the projections

$$Q_m^k = \mathrm{pr}_{\bar{\mathcal{T}}^k(V)} \circ Q_m \colon \bar{\mathcal{T}}^m(V) \to \bar{\mathcal{T}}^k(V) . \tag{4.59}$$

Then, the following proposition holds.

Proposition 4.1.21. *A coderivation Q of $\bar{\mathcal{T}}(V)$, respectively $\bar{S}(V)$, is uniquely determined by the collection Q_i^1 , $i \in \mathbb{N}^+$ by the formula*

$$\begin{aligned} Q_m(x_1 \otimes \cdots \otimes x_m) &= Q_m^1(x_1 \otimes \cdots \otimes x_m) + \\ \sum_{i=1}^{m-1} \sum_{\sigma \in \mathrm{Sh}(i,m-i)} &\epsilon(\sigma) Q_i^1\left(x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(i)}\right) \otimes x_{\sigma(i+1)} \otimes \cdots \otimes x_{\sigma(m)} , \end{aligned} \tag{4.60}$$

Proof. See lemma 2.4 in [12] or appendix A in [69].

Definition 4.1.22. A morphism between connected coalgebras $(C_1, \Delta_1, \epsilon_1, 1_{C_1})$ and $(C_2, \Delta_2, \epsilon_2, 1_{C_2})$ is a degree zero linear map $F\colon C_1 \to C_2$ satisfying

$$\Delta_2 \circ F = (F \otimes F) \circ \Delta , \tag{4.61}$$

$$\epsilon_1 = \epsilon_2 \circ F \,. \tag{4.62}$$

We have that, since $\mathbb{R}$ is a field, F automatically preserves the coaugmentations. Therefore, it can be uniquely determined by its restriction to $\bar{C}$. Hence, morphisms between such coalgebras correspond to degree zero linear maps $F : \bar{C}_1 \to \bar{C}_2$ that satisfy $\bar{\Delta}_2 \circ F = (F \otimes F) \circ \bar{\Delta}_1$. The condition (4.62) is just the dual of the condition

$$\phi\,(1_{X_1}) = 1_{X_2} \,, \tag{4.63}$$

where $X_i\,,\, i = 1, 2$

Proposition 4.1.23. *Let $(C, \Delta, \epsilon, 1_C)$ be a connected coalgebra and let $f : \bar{C} \to V$ be a degree zero linear map from $\bar{C} = \ker \epsilon$ to a graded vector space V. Then, there exists a unique morphism of connected coalgebras $F : C \to S\,(V)$ such that $\mathrm{pr}_V \circ F|_{\bar{C}} = f$, where $\mathrm{pr}_V : S\,(V) \to V$ is the corresponding proyection.*

Proof. See references [68] and [70].

Since it will be important in chapter 5 for the study of L_∞ algebras, let us consider the particular case when $\bar{C} = \bar{S}\,(V)$. We define then the degree zero linear map

$$F^1 : \bar{S}\,(V_1) \to V_2 \,, \tag{4.64}$$

where V_1 and V_2 are graded vector spaces. Additionally, we define the restrictions F^1_k as follows

$$F^1_k = F^1|_{\bar{S}^k(V_1)} \,, \tag{4.65}$$

and hence

$$F^1 = \sum_k^\infty F^1_k \,. \tag{4.66}$$

Then, the following corollary of proposition 4.1.23 holds

Corollary 4.1.24. *Let V_1 and V_2 be graded vector spaces and let $F^1 : \bar{S}\,(V_1) \to V_2$ be a degree zero linear map. There exists then a unique morphism of coalgebras*

$$F : S\,(V_1) \to S\,(V_2) \,, \tag{4.67}$$

such that it satisfies $\mathrm{pr}_{V_2} \circ F|_{\bar{S}(V_2)} = F^1$, where pr_{V_2} is the corresponding projection to V_2.

The construction of the coalgebra morphism F for this particular case can be found in proposition A.2 of [69].

4.2 Categories and complexes

Definition 4.2.1. A category C is:

- A collection of objects $Ob(C)$. Writing $X \in Ob(C)$ denotes that X is an object in the category C.
- For any pair of objects $X, Y \in Ob(C)$ there is attached a set $Hom_C(X, Y)$ which is the set of morphisms between these two objects. We denote a morphism $\phi \in Hom_C(X, y)$ by an arrow $\phi \colon X \to Y$.

- For any three objects X, Y, Z we have composition of morphisms:

$$\begin{aligned} Hom_C(X,Y) \times Hom_C(Y,Z) &\rightarrow Hom_C(X,Z) \\ (\phi, \psi) &\mapsto \psi \circ \phi . \end{aligned} \tag{4.68}$$

  We require that this composition is associative. Here we use that the morphisms between objects form a set and in a set we know what equality between elements means.

- For any object $X \in Ob(C)$ we have a distinguished element $Id_X \in Hom_C(X,X)$, which is an identity on both sides under the composition.

Remark 4.2.2. The definition of category given above is not fully general, since we assume that for every two objects $X, Y \in Ob(C)$, $Hom_C(X,Y)$ is a set. The class of categories we have defined are usually known in the literature as **locally small categories**.

Remark 4.2.3. Given a category C, we are not assuming that the collection $Ob(C)$ is a set. This means that we do not have the notion of equality of two objects. Although we cannot say when two objects in C are equal, we can say when they are isomorphic: two objects $X, Y \in Ob(C)$ are isomorphic if we can find two arrows $\phi\colon X \to Y$ and $\psi\colon Y \to X$ such that $Id_X = \psi \circ \phi$ and $Id_Y = \phi \circ \psi$. However, in general it may be possible to find many such isomorphisms and hence we may have many choices to identify them.

Categories are ubiquitous in mathematics. We present now three simple examples.

Example 4.2.4. The category Set of of sets where the arrows are arbitrary maps of sets.

Example 4.2.5. The category $Vect^d_{\mathbb{K}}$ of d-dimensional vector spaces over a given field $\mathbb{K}$ where the arrows are $\mathbb{K}$-linear maps.

Example 4.2.6. The category Mod_A of modules over a ring A where the arrows consist of A-linear maps. Likewise, we can also define the category of abelian groups Ab, where the morphisms are homomorphisms of groups.

Example 4.2.7. The category Top of topological spaces where the arrows are given by continuous maps.

In the category $Vec^d_{\mathbb{K}}$ all the objects are isomorphic. Since the isomorphism is not unique, we cannot identify them and therefore we can not define a notion of equality. However, if we consider the category of framed d-dimensional vector spaces over $\mathbb{K}$, namely vector spaces equipped with an indexed basis, then the isomorphism between two d-dimensional framed vector spaces is unique, and hence we can define the notion of *equality*.

Remark 4.2.8. Note that the axioms of a category C do not imply that the elements in $Hom_C(X,Y)$ are actual maps between sets (preserving the appropriate additional structure).

Given a cateogry C it is sometimes convenient to define the *opposite category* C^{opp} as a category whose objects are the same as those of C and such that, for every $X, Y \in Ob(C^{opp})$ we have:

$$Hom_{C^{opp}}(X,Y) = Hom_C(Y,X) . \tag{4.69}$$

We introduce now the notion of functor of categories, which loosely speaking can be understood as an extension of the notion of maps between sets to the realm of categories.

Definition 4.2.9. Let C_1 and C_2 be categories. A **covariant functor** $F\colon C_1 \to C_2$ from C_1 to C_2 is a rule that:

- Assigns to every object $X \in Ob(C_1)$ an object $F(X) \in Ob(C_2)$.

- For every two objects $X, Y \in Ob(C_1)$ it induces a map:

$$F_{X,Y} \colon Hom_{C_1}(X,Y) \to Hom_{C_2}(F(X), F(Y)), \tag{4.70}$$

which respects the identity elements and composition.

A covariant functor preserves the direction of the arrows. On the other hand, a contravariant functor is defined as a covariant functor which reverses the direction of the arrows.

Any object $X \in Ob(C)$ in a category C defines a functor h_X from C to the category Set by assigning to every object $Y \in Ob(C)$ the object $h_X(Y) = Hom_C(X,Y) \in Ob(Sets)$. Likewise, to every arrow $\phi \colon Y \to Z$ h_X assigns the arrow:

$$h_X(\phi) \colon Hom_C(X,Y) \to Hom_C(X,Z), \tag{4.71}$$

by composition. We present now two classical examples of functors.

Example 4.2.10. There is a contravariant functor from the category of d-dimensional $\mathbb{K}$-vector spaces $Vect^d_{\mathbb{K}}$ into itself. It sends every vector space $V \in Ob(Vect^d_{\mathbb{K}})$ to its dual vector space $V^* \in Hom_{\mathbb{K}}(V, \mathbb{K})$ and it sends every linear map to its dual linear map. Since the dual of a linear maps goes in the opposite direction, this is a contravariant functor.

Example 4.2.11. A very interesting example of functor is given by the singular Homology functor $F \colon Top \to Ab$, from the category of topological spaces Top to the category of abelian groups Ab, with arrows given by group morphisms. To every topological space $X \in Top$, $F(X) \in Ab$ is the singular Homology group $H_*(X, \mathbb{Z})$ of X. A continuous map $f \colon X \to Y$ for $X, Y \in Ob(Top)$ induces an homomorphism $f_* \colon F(X) = H_*(X, \mathbb{Z}) \to F(Y) = H_*(Y, \mathbb{Z})$ and hence the assignment to the singular Homology group to a topological space is indeed a functor.

Definition 4.2.12. Let C_1 and C_2 be categories and let $F \colon C_1 \to C_2$ and $H \colon C_1 \to C_2$ be functors. A **natural transformation** $\mathcal{T} \colon F \to H$ from F to H which assigns to every object $X \in Ob(C_1)$ a morphism $\mathcal{T}(X) \colon F(X) \to H(X)$ such that for every morphism $f \colon X \to Y$, where $Y \in Ob(C_1)$, the following diagram commutes:

$$\begin{tikzcd}
F(X) \arrow[r, "\mathcal{T}(X)"] \arrow[d, "F(f)"'] & H(X) \arrow[d, "H(f)"] \\
F(Y) \arrow[r, "\mathcal{T}(Y)"] & H(Y)
\end{tikzcd}$$

Given a natural transformation $\mathcal{T} \colon F \to H$, if there is a natural inverse transformation $\mathcal{T}^{-1}$ then we say that $\mathcal{T}$ is a natural isomorphism and hence for any object $X \in Ob(C_1)$ we have that $F(X)$ is isomorphic to $H(X)$. This isomorphism is canonical for every object $X \in Ob(C_1)$ and it involves no choices.

Let C be a category and let $F \colon C \to Set$ be a functor into the category of sets and maps. We may wonder about the possibility of *representing* the category C in terms of F and *set*. This gives rise to the concept of *representable functor*.

Definition 4.2.13. Let C be a category and let Set be the category of sets and maps. For each object $X \in C$ we define the *hom functor* $Hom(X, \cdot) \colon C \to Set$ to be the functor that maps an objects $Y \in C$ to the set $Hom(X,Y)$.

Definition 4.2.14. A functor $F\colon C \to Set$ is said to be *representable* if it is naturally isomorphic to the hom functor $Hom(X,-)$ for some object $X \in C$. In that case, a representation of $F\colon C \to Set$ is a pair (X, T), where $T\colon Hom(X,-) \to F$ is a natural isomorphism.

Definition 4.2.15. An object $0 \in Ob(C)$ in a category C is called the **zero object** if, for all $X \in Ob(C)$, there are two unique arrows $0 \to X$ and $X \to 0$. For $X, Y \in Ob(C)$, an arrow $X \to Y$ is called the **zero morphism** if it factorizes $X \to 0 \to Y$.

Definition 4.2.16. Let C be a category and let $f\colon X \to Y$ be an arrow, where $X, Y \in Ob(C)$. We say that $k\colon K \to X$ is the **kernel** of f if $f \circ k = 0$ and, for any other morphism $k'\colon K' \to X$ such that $f \circ k' = 0$ there exists a morphism $h\colon K' \to K$ such that $k' = k \circ h$.

The **cokernel** of $f\colon X \to Y$ is a dual element to the kernel, namely it is a morphism $k^*\colon Y \to K^*$ such that $k^* \circ f = 0$ and for any other morphism $k^{*\prime}\colon Y \to K^{*\prime}$ satisfying $k^{*\prime} \circ f = 0$ there exists a morphism $h\colon K^* \to K^{*\prime}$ such that $k^{*\prime} = h \circ k^*$.

4.2.1 Homological algebra

The chain complex is the basic structure of homological algebra. It is defined as follows

Definition 4.2.17. A chain complex C is a sequence $(C_\bullet, d_\bullet)$ of abelian groups and group homomorphisms

$$C_\bullet : \cdots \longrightarrow C_{n+1} \stackrel{d_{n+1}}{\longrightarrow} C_n \stackrel{d_n}{\longrightarrow} C_{n-1} \stackrel{d_{n-1}}{\longrightarrow} \cdots, \tag{4.72}$$

such that

$$d_n \circ d_{n+1} = 0\,. \tag{4.73}$$

The abelian groups C_i, $i \in \mathbb{Z}$ are the so-called i-chains and the homomorphisms d_i are called the so-called boundary maps.

Analogously, one can define a co-complex by a simple relabelling of the i-chains and the homomorphisms d_i as follows

Definition 4.2.18. A cochain C is a sequence $(C^\bullet, d^\bullet)$ of abelian groups and group homomorphisms

$$C^\bullet : \cdots \longrightarrow C^{n-1} \stackrel{d^{n-1}}{\longrightarrow} C^n \stackrel{d^n}{\longrightarrow} C^{n+1} \stackrel{d^{n+1}}{\longrightarrow} \cdots, \tag{4.74}$$

such that

$$d^n \circ d^{n-1} = 0\,. \tag{4.75}$$

The abelian groups C^i, $i \in \mathbb{Z}$ are the so-called i-co-chains and the homomorphisms d^i are called the so-called coboundary maps.

In this letter we will use the cochain notation. From equation (4.75) we immediately deduce that

$$B^i = \operatorname{Im} d^{i-1} \subseteq Z^i = \operatorname{Ker} d^i\,, \qquad i \in \mathbb{Z}\,, \tag{4.76}$$

where Ker and Im respectively denote the kernel and image of the given map. Since subgroups of abelian groups are normal, we can define a new group by taking the corresponding quotient. We define this way the ith-cohomology group as follows

$$H^i(C) = \frac{Z^i}{B^i}\,, \qquad i \in \mathbb{Z}\,. \tag{4.77}$$

A cochain is called an exact sequence if all its cohomology groups are zero. The cochain groups C^i may be endowed with extra structure; for example, they may be vector spaces or modules over a fixed ring $\mathbb{K}$. In that case, the coboundary operators must preserve the extra structure if it exists; for instance, they must be linear maps or homomorphisms of $\mathbb{K}$-modules.

Let $C = (C^\bullet, d_C^\bullet)$ and $D = (D^\bullet, d_D^\bullet)$ be cochain complexes. A morphism $F : C^\bullet \to D^\bullet$ between C and D is a family of homomorphisms of abelian groups $F^i : C^i \to D^i$, $i \in \mathbb{Z}$ that commute with the co-boundary operators, that is

$$F^{i+1} \circ d_C^i = d_D^i \circ F^i \,, \qquad i \in \mathbb{Z} \,. \tag{4.78}$$

A morphism of chain complexes induces a morphism F_H of their homology groups, consisting of the homomorphisms F_H^i, $i \in \mathbb{Z}$ defined by

$$F_H^i([x]) = \left[F^i(x)\right] , \qquad \forall [x] \in H^i(C) \,,\ i \in \mathbb{Z} \,. \tag{4.79}$$

A morphism F such that F_H is an isomorphism, that is, that induces an isomorphism in cohomology, is called a *quasi-isomorphism*. The prominent example of cochain that will appear in this letter is the L_∞-algebra, which we will introduce in chapter 5. For L_∞-algebras we will need to introduce a less restrictive concept of morphism, which is adapted to the structure present in L_∞-algebras.

Suppose that there exists a map $f : X \to Y$ between two objects X and Y. Then, Homological algebra studies the relation, induced by the map f, between chain complexes (or co-complexes) associated with X and Y and their homology (or cohomology).

Example 4.2.19. Given a manifold $\mathcal{M}$ we can construct the complex

$$C^i \equiv \Omega^i(\mathcal{M}) \,, \tag{4.80}$$

where $\Omega^i(\mathcal{M})$ denotes the set of i-forms on $\mathcal{M}$[6]. The co-boundary operator is the exterior derivative $d^i : \Omega^i(\mathcal{M}) \to \Omega^{i+1}(\mathcal{M})$. The corresponding cohomology is the *de-Rham cohomology*

$$H^i(\mathcal{M}) = \frac{\operatorname{Ker} d^i}{\operatorname{Im} d^{i-1}} \,. \tag{4.81}$$

The de-Rahm cohomology groups give important information about the manifold where it is defined. For instance, the zero cohomology group $H^0(\mathcal{M})$ of any differentiable manifold $\mathcal{M}$ is given by

$$H^0(\mathcal{M}) \simeq \mathbb{R}^n \,, \tag{4.82}$$

where n is the number of connected components of $\mathcal{M}$. This can be easily seen from the fact that any function $f \in C^\infty(\mathcal{M})$ such that $df = 0$ is constant on each of the connected component of $\mathcal{M}$. Therefore, the dimension of the $H^0(\mathcal{M})$ gives the number of connected componentes of $\mathcal{M}$.

Example 4.2.20. **Lie Algebra Cohomology**. Let $\mathfrak{g}$ be a Lie algebra. We define the following cochain complex $C = (C^\bullet, \delta^\bullet)$

$$C^i \equiv \Lambda^i \mathfrak{g}^* \,, \tag{4.83}$$

with coboundary operator $\delta^k : C^k \to C^{k+1}$ given by

$$\delta^k c(x_1, \cdots, x_k) = \sum_{1 \leq i < j \leq k} (-1)^{i+j} c([x_i, x_j], x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_k) \,, \tag{4.84}$$

[6]See section 2.1 for more details.

for all $x_1, \cdots, x_k \in \mathfrak{g}$. Notice that we interpret an element $c \in C^i$ as an alternating i-linear operator $c : \mathfrak{g}^{\times k} \to \mathbb{R}$. δ is in fact a coboundary operator; it can be checked that $\delta^2 = 0$. We can introduce now the *Lie algebra cohomology groups*, or *Chevalley cohomology groups*, of $\mathfrak{g}$ using (4.77)

$$H^i(\mathfrak{g}, \mathbb{R}) \equiv \frac{\mathrm{Ker}\,\delta^i}{\mathrm{Im}\,\delta^{i-1}}\,. \tag{4.85}$$

which in fact contain the same information as the de-Rham cohomology groups of G, in the following sense

Theorem 4.2.21. *If* $\mathfrak{g}$ *is the Lie algebra of compact connected Lie group* G*, then*

$$H^i(\mathfrak{g}, \mathbb{R}) = H^i_{\text{de Rham}}(G)\ . \tag{4.86}$$

Chapter 5

L_∞-algebras

In this section we review L_∞-algebras and explicitly describe general L_∞-morphisms. An L_∞-algebra structure on graded vector space L can be defined to be a collection of skew-symmetric maps $\{l_k\colon L^{\otimes k} \to L\}_{k=1}^{\infty}$ with $|l|_k = k-2$ which satisfy a rather complicated generalization of the Jacobi identity. We will therefore start with a more elegant description, given in terms of coalgebras, and prove its equivalence to the previous characterization.

5.1 Basic definitions

Definition 5.1.1. An $L_\infty[1]$-structure on a graded vector space M is a choice of degree one codifferential Q on the coalgebra

$$C(M) = \bar{S}\,(M)\;. \tag{5.1}$$

Theorem 5.1.2. *An $L_\infty[1]$-structure on a graded vector space M, that is, a choice of degree one codiferential Q on $\bar{S}(M)$, uniquely determines a family of degree one linear maps*

$$\Big(m_k : \bar{S}^k(M) \to M\Big)_{k\in\mathbb{N}^+}\;, \tag{5.2}$$

such that

$$\sum_{r+s=k}\sum_{\sigma\in \mathrm{Sh}(r,s)} \epsilon(\sigma) m_{(s+1)}\left(m_r\left(x_{\sigma(1)}\otimes\cdots\otimes x_{\sigma(r)}\right)\otimes x_{\sigma(r+1)}\otimes\cdots\otimes x_{\sigma(k)}\right) = 0\,. \tag{5.3}$$

where $\epsilon(\sigma) = \epsilon(\sigma; x_1,\dots,x_r)$ and $x_1,\dots,x_k \in M$. Conversely, any such family $(m_k)_{k\in\mathbb{N}^+}$ of degree one linear maps uniquely determines a degree one codifferential Q on $\bar{S}(M)$.

Proof. Given a codifferential Q on $\bar{S}(M)$, consider the restrictions

$$Q_k = Q|_{\bar{S}^k(M)}\colon \bar{S}^k(M) \to \bar{S}(M), \quad 1 \le k < \infty\,, \tag{5.4}$$

so that $Q = \sum_k^\infty Q_k$, and also the projections

$$Q_m^k = \mathrm{pr}_{\bar{S}^k(M)} \circ Q_m\colon \bar{S}^m(M) \to \bar{S}^k(M)\,. \tag{5.5}$$

It follows from proposition 4.1.21 that Q can be uniquely determined by the collection of maps

$$Q_k^1 = \mathrm{pr}_M \circ Q_k\colon \bar{S}^k(M) \to M, \quad k\ge 1\,. \tag{5.6}$$

The complete coderivation Q can be written as

$$Q_m\left(x_1 \odot \cdots \odot x_m\right) = Q_m^1\left(x_1 \odot \cdots \odot x_m\right) + \tag{5.7}$$

$$\sum_{i=1}^{m-1} \sum_{\sigma \in \mathrm{Sh}(i,m-i)} \epsilon(\sigma)\, Q_i^1\left(x_{\sigma(1)} \odot \cdots \odot x_{\sigma(i)}\right) \odot x_{\sigma(i+1)} \odot \cdots \odot x_{\sigma(m)}\,, \tag{5.8}$$

for any $x_i \in M$. Defining now the maps $(m_k)_{k\in\mathbb{N}^+}$ as follows

$$\left(m_k = Q_k^1 : \bar{S}^k(M) \to M\right)_{k\in\mathbb{N}^+}\,, \tag{5.9}$$

the condition $Q \circ Q = 0$ is equivalent to the generalized Jacobi identity (5.3) for the collection $(m_k)_{k\in\mathbb{N}^+}$. In particular, it implies that l_1 is degree one differential on L. On the other hand, let us assume that $\bar{S}(M)$ is equipped with set of degree one maps

$$\left(m_k : \bar{S}^k(M) \to M\right)_{k\in\mathbb{N}^+}\,, \tag{5.10}$$

obeying equation (5.3). Taking the $(m_k)_{k\in\mathbb{N}^+}$ as the $(Q_k^1)_{k\in\mathbb{N}^+}$ components in Lemma 2.4 in [12] we conclude that there exists a unique codifferential Q in $\bar{S}(M)$ such that its $(Q_k^1)_{k\in\mathbb{N}^+}$ restrictions are given by the $(m_k)_{k\in\mathbb{N}^+}$ maps.

□

Hence, an $L_\infty[1]$-structure on a graded vector space M can be equivalently defined in terms of a degree one coderivation Q on $C(M)$ or in terms of a family of morphisms $\left(m_k : S^k(M) \to M\right)_{k\in\mathbb{N}^+}$ obeying (5.3). An L_∞-structure is related to an $L_\infty[1]$-structure by a degree shift in M. In particular, an L_∞-structure on a graded vector space L is nothing but an $L_\infty[1]$-structure on the graded vector space $M = s^{-1}L$.

Definition 5.1.3. An L_∞-structure on a graded vector space L is an $L_\infty[1]$-structure on the graded vector space $s^{-1}L$.

An equivalent, more practical, definition is the following

Definition 5.1.4 ([12])**.** An L_∞-algebra is a graded vector space L together with a collection

$$\left\{l_k\colon L^{\otimes k} \to L | 1 \leq k < \infty\right\} \tag{5.11}$$

of graded skew-symmetric linear maps with $|l_k| = 2 - k$ such that the following identity is satisfied for $1 \leq m < \infty$

$$\sum_{\substack{i+j=m+1\,,\\ \sigma\in\mathrm{Sh}(i,m-i)}} (-1)^{\sigma}\, \epsilon(\sigma)\, (-1)^{i(j-1)}\, l_j\left(l_i(x_{\sigma(1)}, \ldots, x_{\sigma(i)}), x_{\sigma(i+1)}\,, \ldots, x_{\sigma(m)}\right) = 0\,. \tag{5.12}$$

Proof. Let us prove the equivalence of both definitions. If we consider an $L_\infty[1]$ structure on $s^{-1}L$ then condition (5.3) can be written as

$$\sum_{r+s=k} \sum_{\sigma\in\mathrm{Sh}(r,s)} \epsilon(\sigma) m_{(s+1)}\left(m_r\left(s^{-1}x_{\sigma(1)} \otimes \cdots \otimes s^{-1}x_{\sigma(r)}\right) \otimes s^{-1}x_{\sigma(r+1)} \otimes \cdots \otimes s^{-1}x_{\sigma(k)}\right) = 0\,. \tag{5.13}$$

where $\epsilon(\sigma) = \epsilon\left(\sigma; s^{-1}x_1, \ldots, s^{-1}x_k\right)$ and $x_1, \ldots, x_k \in L$. Using equation (4.29) it can be proven that

$$m_r\left(s^{-1}x_{\sigma(1)}\otimes\cdots\otimes s^{-1}x_{\sigma(r)}\right)=(-1)^{\sum_{i=1}^{r}(r-i)|x_{\sigma(i)}|}m_r\circ s^{-r}\left(x_{\sigma(1)}\otimes\cdots\otimes x_{\sigma(r)}\right)\,. \tag{5.14}$$

Using now equation (5.14) together with the following commutative diagram

$$\begin{tikzcd}
\Lambda^k(L) \arrow[r, "l_k"] \arrow[d, "s^{-k}"'] & L \arrow[d, "s^{-1}"] \\
S^k(s^{-1}L) \arrow[r, "m_k"] & s^{-1}L
\end{tikzcd}$$

we obtain

$$m_r\circ s^{-r}\left(x_{\sigma(1)}\otimes\cdots\otimes x_{\sigma(r)}\right)=s^{-1}\circ l_r\left(x_{\sigma(1)}\otimes\cdots\otimes x_{\sigma(r)}\right)\,. \tag{5.15}$$

Therefore

$$\begin{aligned}
&m_{(s+1)}\left(m_r\left(s^{-1}x_{\sigma(1)}\otimes\cdots\otimes s^{-1}x_{\sigma(r)}\right)\otimes s^{-1}x_{\sigma(r+1)}\otimes\cdots\otimes s^{-1}x_{\sigma(k)}\right)=\\
&(-1)^{\sum_{i=1}^{r}(r-i)|x_{\sigma(i)}|+\sum_{i=1}^{k-r+1}(k-r+1-i)|\tilde{x}_i|}s^{-1}\circ l_{(s+1)}\left(l_r\left(x_{\sigma(1)}\otimes\cdots\otimes x_{\sigma(r)}\right)\otimes x_{\sigma(r+1)}\otimes\cdots\otimes x_{\sigma(k)}\right)
\end{aligned} \tag{5.16}$$

where

$$\tilde{x}_1=l_r\left(x_{\sigma(1)},\ldots,x_{\sigma(r)}\right)\,,\quad \tilde{x}_i=x_{\sigma(r-1+i)}\,,\quad 2\leq i\leq k-r+1\,. \tag{5.17}$$

Since

$$|l_r\left(x_{\sigma(1)},\ldots,x_{\sigma(r)}\right)|=2-r+\sum_{i=1}^{r}|x_{\sigma(i)}|\,, \tag{5.18}$$

we have

$$\sum_{i=1}^{k-r+1}|\tilde{x}_i|=2-r+\sum_{i=1}^{k}|x_{\sigma(i)}|\,,\quad (-1)^{\sum_{i=1}^{r}(r-i)|x_{\sigma(i)}|+\sum_{i=1}^{k-r+1}(k-r+1-i)|\tilde{x}_i|}=(-1)^{(k-r)r+\sum_{i=1}^{k}(k-i)|x_{\sigma(i)}|}\,. \tag{5.19}$$

Finally, using equation (4.34) together with equation (5.19) in equation (5.3), we obtain equation (5.12). □

Therefore, any L_∞-algebra (L,l_k) corresponds to a certain kind of graded co-algebra $C(s^{-1}L)$ equipped with a co-derivation Q which satisfies the identity

$$Q\circ Q=0\,. \tag{5.20}$$

As we have seen, this identity is the origin of equation (5.12). It is easy to see that for small values of m that equation (5.12) is a *generalized Jacobi identity* for the multi-brackets $\{l_k\}$. For $k=1$, it implies that the degree one linear map l_1 satisfies

$$l_1 \circ l_1 = 0 \tag{5.21}$$

and hence every L_∞-algebra (L, l_k) has an underlying cochain complex $(L, d = l_1)$. For $k=2$ we have that $[\cdot,\cdot] = l_2$ is a degree zero linear map that satisfies

$$d\,[x_1, x_2] = [dx_1, x_2] + (-1)^{|x_1|}\,[x_1, dx_2]\ . \tag{5.22}$$

Hence l_2 can be interpreted as a bracket, which is skew symmetric

$$[x_1, x_2] = -(-1)^{|x_1||x_2|}\,[x_2, x_1]\ , \tag{5.23}$$

but does not satisfy the usual Jacobi identity.

Definition 5.1.5. A Lie n-algebra is a L_∞-algebra $(L, \{l_k\})$ such that the corresponding graded vector space L is concentrated in degrees $0, -1, \ldots, 1-n$.

Notice that if $(L, \{l_k\})$ is a Lie n-algebra, simply by degree counting then $l_k = 0$ for $k > n+1$. Therefore, a Lie 1-algebra is nothing but a Lie algebra.

5.2 L_∞-morphisms

The notion of L_∞-morphism will be essential in this work. We now give [12] a naive definition of what could be an $L_\infty - morphism$.

Definition 5.2.1. If (L^1, l^1_k) and (L^2, l^2_k) be L_∞-algebras then a degree zero linear map $f\colon L^1 \to L^2$ is a *strict L_∞-morphism* if and only if the following holds

$$l^2_k \circ f^{\otimes k} = f \circ l^1_k \quad \forall k \geq 1\,. \tag{5.24}$$

The definition above however does not reflect the higher structure which resides within the theory in a natural way. Actually, there is a better definition, see Remark 5.3 of [12], which uses the previously mentioned relationship between L_∞-algebras and differential graded coalgebras. This turns out to give to the collection of morphisms between two L_∞-algebras the structure of a simplicial set, see reference [71], which therefore permits to consider homotopies among morphisms, homotopies among homotopies *et cetera*. As we did when we defined L_∞-algebras, we define first a morphism of $L_\infty[1]$-algebras. The corresponding definition for L_∞-algebras can then be obtained by a degree shift in L.

Definition 5.2.2. An $L_\infty[1]$-morphism between $L_\infty[1]$-algebras (M^1, m^1_k) and (M^2, m^2_k) is a morphism $F[1]\colon\ (C(M^1), Q^1) \to (C(M^2), Q^2)$ between the corresponding underlying differential graded coalgebras. $F[1]$ is thus a morphism between the graded coalgebras $C(M^1)$ and $C(M^2)$ such that

$$F[1] \circ Q^1 = Q^2 \circ F[1]\,. \tag{5.25}$$

As it turns out, an $L_\infty[1]$-morphism $F[1]$ between (M^1, m^1_k) and (M^2, m^2_k) corresponds to an infinite collection of symmetric, degree zero, ‘structure maps’

$$F[1] = \left(f_k[1]\colon S^k\left(M^1\right) \to M^2 \quad 1 \leq k < \infty\right)\,, \tag{5.26}$$

and such that a given compatibility relation with the multi-brackets must be satisfied. More precisely, the following proposition holds.

Proposition 5.2.3. *Let (M^1, m_k^1) and (M^2, m_k^2) be $L_\infty[1]$-algebras. A morphism from (M^1, m_k^1) to (M^2, m_k^2) is a family of morphism*

$$F[1] = \left(f_k[1] \colon S^k\left(M^1\right) \to M^2 \quad 1 \leq k < \infty\right)\,, \tag{5.27}$$

such that

$$\sum_{r+s=k}\sum_{\sigma\in \mathrm{Sh}(r,s)} \epsilon\left(\sigma; x_1, \ldots, x_k\right) f_{s+1}[1]\left(m_r\left(x_{\sigma(1)}, \ldots, x_{\sigma(r)}\right), x_{\sigma(r+1)}, \ldots, x_{\sigma(k)}\right) =$$
$$\sum_{l=1}^{k}\sum_{j_1+\cdots+j_l=k}\sum_{\tau\in\Sigma_k} \frac{\epsilon\left(\tau; x_1, \ldots, x_k\right)}{l! j_1! \ldots j_l!} n_l\left(f_{j_1}[1]\left(x_{\sigma(\tilde{k}_1+1)}, \ldots, x_{\sigma(\tilde{k}_1+j_1)}\right), \ldots, f_{j_l}[1]\left(x_{\sigma(\tilde{k}_l+1)}, \ldots, x_{\sigma(\tilde{k}_l+j_l)}\right)\right) \tag{5.28}$$

where $\tilde{k}_1 = 0$ and $\tilde{k}_s = \sum_{i=1}^{s-1} j_i\,,\; 1 < s \leq l$.

Proof. See chapter 2 of [72].

The corresponding notion of morphism of L_∞-algebras goes as follows

Definition 5.2.4. An L_∞-morphism F between L_∞-algebras (L^1, l_k^1) and (L^2, l_k^2) is a morphism $F\colon\ (C(s^{-1}L^1), Q^1$ $(C(s^{-1}L^2), Q^2)$ between their corresponding differential graded coalgebras. F is hence a morphism between the graded coalgebras $C(s^{-1}L^1)$ and $C(s^{-1}L^2)$ such that

$$F \circ Q^1 = Q^2 \circ F\,. \tag{5.29}$$

As in the $L_\infty[1]$ case, the notion of L_∞-morphism corresponds to an infinite family of maps, which in this case are skew-symmetric structure maps

$$F = \left(f_k \colon \Lambda^k L^1 \to L^2 \quad 1 \leq k < \infty\right)\,, \tag{5.30}$$

where $|f_k| = 1 - k$. Here again the family of maps have to satisfy a somewhat complicated compatibility relation involving the multi-brackets. Such compatibility relation can be obtained from equation (5.28) by means of the following commutative diagram

$$\begin{tikzcd}
\Lambda^k\left(L^1\right) \arrow[r, "f_k"] \arrow[d, "s^{-k}"'] & L^2 \arrow[d, "s^{-1}"] \\
S^k\left(s^{-1}L^1\right) \arrow[r, "{f[1]_k}"] & s^{-1}L^2
\end{tikzcd}$$

by performing the corresponding degree shift in L^1 and L^2.

We see that, particular, the degree zero map f_1 is a morphism between the corresponding complexes (L^1, l_1^1) and (L^2, l_1^2)

$$f_1 \circ l_1^1 = l_1^2 \circ f_1\,. \tag{5.31}$$

As it happens for $L_\infty[1]$-morphisms, the compatibility relation between the family of maps $(f_i)_{i\geq 1}$ and the multibrackets precisely corresponds in the language of coalgebras to equation (5.29). It can be easily seen that strict morphisms as defined in 5.2.1 correspond to the case given by $f_i = 0$ for $i \geq 2$. L_∞-morphisms can be composed in the standard sense, and therefore it is possible to consider the category of L_∞-algebras without explicit use the higher structure present in L_∞-morphisms.

We define now the notion of L_∞-*quasi-isomorphism*. This definition naturally reflexts the homotopical structure that exists between morphisms.

Definition 5.2.5. Let $(f_k)\colon (L^1, l^1_k) \to (L^2, l^2_k)$ be an L_∞-algebra morphisim. Then we say that $(f_k)_{k\geq 1}$ is an L_∞-quasi-isomorphism if and only if the corresponding morphism of complexes

$$f_1\colon (L^1, l^1_1) \to (L^2, l^2_1) \tag{5.32}$$

induces an isomorphism on the cohomology of the underlying complexes

$$H^\bullet(f_1) : H^\bullet\left(L^1\right) \xrightarrow{\cong} H^\bullet\left(L^2\right) . \tag{5.33}$$

5.2.1 Morphisms from Lie algebras to L_∞-algebras

Since it will be useful in chapter 6, we will consider L_∞-algebra morphisms whose sources are simply Lie algebras $(\mathfrak{g}, [\cdot,\cdot])$. In that case the conditions that the components of the morphism must satisfy are extremely simplified and the resulting expression can be indeed used for practical purposes. We will assume also that the image (L, l_k) of the L_∞-algebra morphism is a Lie-n algebra such that

$$\forall\ i \geq 2 \qquad l_i(x_1, \ldots, x_i) = 0 \quad \text{whenever} \quad \sum_{k=1}^{i} |x_k| < 0 . \tag{5.34}$$

This is indeed the relevant case for Lie-n algebras arising from n-plectic manifolds, as we will see in chapter 6. The relevant proposition is then the following

Proposition 5.2.6. *Let $(\mathfrak{g}, [\cdot,\cdot])$ be a Lie algebra and let (L, l_k) is a Lie n-algebra that satisfies the property* (5.34). *Then a collection of n anti-symmetric maps*

$$f_m : \mathfrak{g}^{\otimes m} \to L, \quad |f_m| = 1 - m, \quad 1 \leq m \leq n , \tag{5.35}$$

can be taken to be the components of an L_∞-morphism $(f_k)_{k\geq 1} : \mathfrak{g} \to L$ if and only if $\forall x_i \in \mathfrak{g}$

$$\sum_{1\leq i<j\leq m} (-1)^{i+j+1} f_{m-1}\left([x_i, x_j], x_1, \ldots, \widehat{x_i}, \ldots, \widehat{x_j}, \ldots, x_m\right) \tag{5.36}$$

$$= l_1 f_m(x_1, \ldots, x_m) + l_m\left(f_1(x_1), \ldots, f_1(x_m)\right) . \tag{5.37}$$

for $2 \leq m \leq n$ and

$$\sum_{1\leq i<j\leq n+1} (-1)^{i+j+1} f_n\left([x_i, x_j], x_1, \ldots, \widehat{x_i}, \ldots, \widehat{x_j}, \ldots, x_{n+1}\right) = l_{n+1}\left(f_1(x_1), \ldots, f_1(x_{n+1})\right) . \tag{5.38}$$

Proof. See appendix A.5 of [18] □

Chapter 6

Multisymplectic Geometry

6.1 Multisymplectic manifolds

We will closely follow [18]. For more details about Multisymplectic Geometry the interested reader can consult [1–4, 10].

Definition 6.1.1. A differentiable manifold $\mathcal{M}$ is said to be **n-plectic** or **multisymplectic** if it is equipped with an $(n+1)$-form $\omega \in \Omega^{n+1}(\mathcal{M})$ such that it is both closed:

$$d\omega = 0\,, \tag{6.1}$$

and non-degenerate:

$$\forall\, p \in \mathcal{M} \qquad \forall\, u \in T_p\mathcal{M}, \qquad \iota_u\omega = 0 \Rightarrow u = 0\,. \tag{6.2}$$

If ω is an n-plectic form on $\mathcal{M}$, then we call the pair $(\mathcal{M}, \omega)$ an n-plectic manifold. More generally, if ω is closed, but not necessarily non-degenerate, then we call $(\mathcal{M}, \omega)$ a pre-n-plectic manifold. We will only deal with n-plectic manifolds, although most of our results can be straightforwardly extended to the pre-n-plectic case.

Remark 6.1.2. A 1-plectic manifold is simply a symplectic manifold.

Definition 6.1.3. Given two n-plectic manifolds $(\mathcal{M}_1, \omega_1)$ and $(\mathcal{M}_2, \omega_2)$, a diffeomorphism $F\colon \mathcal{M}_1 \to \mathcal{M}_2$ is said to be a **multisymplectic diffeomorphism** if $F^*\omega_2 = \omega_1$.

Definition 6.1.4. Given an n-plectic manifold $(\mathcal{M}, \omega)$, an $(n-1)$-form $\beta \in \Omega^{n-1}(\mathcal{M})$ is said to be **Hamiltonian** if and only if there exists a vector field $u_\beta \in \mathfrak{X}(\mathcal{M})$ such that:

$$d\beta = -\iota_{u_\beta}\omega\,. \tag{6.3}$$

We say then that u_β is a **Hamiltonian vector field** corresponding to β. We respectively denote by $\Omega^{n-1}_{\mathrm{Ham}}(\mathcal{M})$ and $\mathfrak{X}_{\mathrm{Ham}}(\mathcal{M})$, the set of Hamiltonian $(n-1)$-forms and the set of Hamiltonian vector fields on an n-plectic manifold, which are real vector spaces. Note that, due to the non-degeneracy of ω, for every Hamiltonian form there is a unique Hamiltonian vector field associated.

Definition 6.1.5. A vector field u on a n-plectic manifold $(\mathcal{M}, \omega)$ is a *local Hamiltonian vector field* if and only if

$$\mathcal{L}_u\omega = 0\,, \tag{6.4}$$

We denote by $\mathfrak{X}_{\mathrm{Ham}}(\mathcal{M})$ the vector space of local Hamiltonian vector fields.

Please notice that equation (6.4) is equivalent to

$$di_v\omega = 0\,. \tag{6.5}$$

Hence, for Hamiltonian vector fields $i_u\omega$ is an exact n-form while for locally Hamiltonian vector fields $i_u\omega$ is a closed n-form, which can be always locally written in terms of an exact form, justifying name *local Hamiltonian vector field.* If $H^1_{\mathrm{dR}}(\mathcal{M}) = 0$ both definitions of course coincide.

Definition 6.1.6. Let $(\mathcal{M}, \omega)$ be a n-plectic manifold. Given $\alpha, \beta \in \Omega^{n-1}_{\mathrm{Ham}}(\mathcal{M})$, we define the *bracket* $\{\alpha, \beta\}$ to be the $(n-1)$-form given by

$$\{\alpha, \beta\} = \iota_{u_\beta}\iota_{u_\alpha}\omega\,, \tag{6.6}$$

where u_α and u_β respectively stand for the Hamiltonian vector fields for α and β.

Proposition 6.1.7. *Let $(\mathcal{M}, \omega)$ be an n-plectic manifold and let $u_1, u_2 \in \mathfrak{X}_{\mathrm{Ham}}(\mathcal{M})$ be local Hamiltonian vector fields. Then $[u_1, u_2]$ is a global Hamiltonian vector field with*

$$d\iota_{u_1\wedge u_2}\omega = -\iota_{[u_1,u_2]}\omega\,, \tag{6.7}$$

and thus $\mathfrak{X}_{\mathrm{Ham}}(\mathcal{M})$ and $\mathfrak{X}_{\mathrm{Ham}}(\mathcal{M})$ are Lie subalgebras of $\mathfrak{X}(\mathcal{M})$.

Proof. Let u_1, u_2 be locally Hamiltonian vector fields. Then by equation (2.22),

$$\mathcal{L}_{u_1}\iota_{u_2}\omega = \iota_{[u_1,u_2]}\omega\,. \tag{6.8}$$

Using now equation (2.21),

$$\mathcal{L}_{u_1}\iota_{u_2}\omega = \iota_{u_1}d\iota_{u_2}\omega + d\iota_{u_1}\iota_{u_2}\omega\,. \tag{6.9}$$

However $\iota_{u_1}d\iota_{u_2}\omega = 0$, since $d\iota_{u_2} = \mathcal{L}_{u_2} - \iota_{u_2}d$. □ □

Proposition6.1.7 implies in particular that if u_α and u_β are respectively Hamiltonian vector fields for α and β, then $[u_\alpha, u_\beta]$ is a Hamiltonian vector field for $\{\alpha, \beta\}$. Notice that the bracket defined in 6.1.6 is skew-symmetric but it fails to satisfy the Jacoby identity. In particular we have[1]

$$\{\alpha_1, \{\alpha_2, \alpha_3\}\} - \{\{\alpha_1, \alpha_2\}, \alpha_3\} - \{\alpha_2, \{\alpha_1, \alpha_3\}\} = -d\iota_{v_{\alpha_1}\wedge v_{\alpha_2}\wedge v_{\alpha_3}}\omega\,. \tag{6.10}$$

Therefore, the space $\Omega^{n-1}_{\mathrm{Ham}}(M)$ of Hamiltonian forms equipped with the bracket $\{\cdot,\cdot\}$ is not a Lie algebra unless $n = 1$, which is the well-know symplectic case. Hence, we cannot straightforwardly extend the Poisson structure present in the set of functions on a symplectic manifold to the set of Hamiltonian forms on a multisymplectic manifold. Nonetheless, equation (6.10) shows that $\{\cdot,\cdot\}$ fails to satisfy the Jacobi identity by an exact form, which suggests the existence of an underlying n-Lie algebra structure of which $\Omega^{n-1}_{\mathrm{Ham}}(M)$ would be part of. Roughly speaking, if we identify the interior product of k Hamiltonian vector fields with ω as l_k acting on the corresponding k Hamiltonian forms, then equation 6.10 is the condition that l_2 and l_3 have to obey if they are part of an Lie-n algebra.

We present now a theorem that gives a natural L_∞-structure, in particular a n-Lie algebra structure, on differential forms, extending the bracket $\{\cdot,\cdot\}$ on $\Omega^{n-1}_{\mathrm{Ham}}(\mathcal{M})$. See theorem 5.2 in [11] and theorem 6.7 in [73]. A detailed exposition can be found in [10].

[1] See proposition 3.5 in reference [10].

Theorem 6.1.8. *Let $(\mathcal{M},\omega)$ be an n-plectic manifold . Then there exists a Lie n-algebra $L_\infty(\mathcal{M},\omega) = (L,\{l_k\})$ with underlying graded vector space*

$$L^i = \begin{cases} \Omega^{n-1}_{\text{Ham}}(\mathcal{M}) & k=0, \\ \Omega^{n-1+k}(\mathcal{M}) & 1-n \leq k < 0\,, \end{cases} \tag{6.11}$$

and maps $\left\{l_k : L^{\otimes k} \to L | 1 \leq k < \infty\right\}$ defined as follows

$$l_1(\beta) = d\beta\,, \tag{6.12}$$

if $|\beta| < 0$ and

$$l_k(\beta_1,\ldots,\beta_k) = \begin{cases} \xi(k)\iota\left(u_{\beta_1}\wedge\cdots\wedge u_{\beta_k}\right)\omega & \text{if } |\beta_1\otimes\cdots\otimes\beta_k| = 0, \\ 0 & \text{if } |\beta_1\otimes\cdots\otimes\beta_k| < 0\,, \end{cases} \tag{6.13}$$

for $k > 1$, where u_{β_i} is the Hamiltonian vector field associated to $\beta_i \in \Omega^{n-1}_{\text{Ham}}(\mathcal{M})$ and $\xi(k) = -(-1)^{\frac{k(k+1)}{2}}$.

We can extend definition 6.1.6 and define a k-ary bracket in $L(\mathcal{M},\omega)$ as follows

$$\{x_1,\ldots,x_k\} = l_k(x_1,\ldots,x_k)\,, \qquad x_1,\ldots,x_k \in L(\mathcal{M},\omega)\,. \tag{6.14}$$

Please notice that in the $n=1$ case, the underlying complex is simply the vector space of Hamiltonian functions $C^\infty(\mathcal{M})$. The only non-zero bracket is therefore $l_2 = \{\cdot,\cdot\}$, which is simply a Lie bracket. We thus recover the Lie algebra which underlies the usual Poisson algebra that can be constructed for symplectic manifold. As explained in section 3.2, in that case there is a well-defined surjective Lie algebra morphism

$$\pi : C^\infty(\mathcal{M}) \twoheadrightarrow \mathfrak{X}_{\text{Ham}}(\mathcal{M})\,, \tag{6.15}$$

which send a function to its (unique) Hamiltonian vector field. If $\mathcal{M}$ is connected, it can be shown that π fits in the following short exact sequence

$$0 \to \mathbb{R} \to C^\infty(\mathcal{M}) \xrightarrow{\pi} \mathfrak{X}_{\text{Ham}}(\mathcal{M}) \to 0\,. \tag{6.16}$$

Equation (6.16) is the so-called Kostant-Souriau central extension, see references [74, 75]. The Kostant-Souriau central extension characterizes,up to isomorphism, the Lie algebra of $C^\infty(\mathcal{M})$ as follows: it is the central extension, which can be shown to be unique, given by the symplectic form, evaluated at $p \in \mathcal{M}$. The higher analog of the central extension (6.16) is given by the cochain map

$$\pi : L(\mathcal{M},\omega) \twoheadrightarrow \mathfrak{X}_{\text{Ham}}(\mathcal{M})\,, \tag{6.17}$$

which is trivial en all degrees but zero. In degree zero it assigns to every Hamiltonian form α its unique Hamiltonian vector field. The map π fits hence in the following short exact sequence

$$0 \to \tilde{\Omega} \to L(\mathcal{M}) \xrightarrow{\pi} \mathfrak{X}_{\text{Ham}}(\mathcal{M}) \to 0\,. \tag{6.18}$$

Here $\tilde{\Omega}$ stands for the cocomplex

$$\tilde{\Omega} = C^\infty(\mathcal{M}) \to \Omega^1(\mathcal{M}) \to \cdots \to \Omega^{n-1}_{\text{cl}}(\mathcal{M})\,, \tag{6.19}$$

where $\Omega^{n-1}_{\mathrm{cl}}(\mathcal{M})$ is the set of closed $(n-1)$-forms in $\mathcal{M}$ and the coboundary operator is the de Rahm differential. We introduce a further sequence of operations on L, which turns out to be very handy for the purposes of this note.

Remark 6.1.9. The operations $[\dots]_k$ on L we introduce now are labelled by integers $k \geq 0$, unlike the operations appearing in theorem 6.1.8. The multilinear maps $[\dots]_k$ are closely related to the multibrackets of $L_\infty(M,\omega)$: for $k \geq 1$,

$$[\alpha_1, \dots, \alpha_k]_k = \{\alpha_1, \dots, \alpha_k\}_k - \delta_{k,1} d\alpha_1 \,,$$

where δ denotes the Kronecker delta. In particular, for $k \geq 2$, $[\dots]_k$ and $\{\dots\}_k$ agree, while $[\alpha]_1$ vanishes if $|\alpha| < 0$ and equals $-d\alpha$ when $|\alpha| = 0$. We also have $[1]_0 = -\omega$.

Explicitly, the operations $[\dots]_k$ are given as follows:

Definition 6.1.10. Let (M,ω) be a n-plectic manifold. Let L denote the graded vector space underlying $L_\infty(M,\omega)$.

For all $k \geq 0$, we define the multilinear maps $[\dots]_k \colon L^{\otimes k} \to \Omega^{n+1-k}(M)$ as follows:

$$[\alpha_1, \dots, \alpha_k]_k = \begin{cases} 0 & \text{if } |\alpha_1 \otimes \cdots \otimes \alpha_k| \leq -1, \\ \varsigma(k)\iota(v_{\alpha_1} \wedge \cdots \wedge v_{\alpha_k})\omega & \text{if } |\alpha_1 \otimes \cdots \otimes \alpha_k| = 0, \end{cases}$$

6.1.1 Homotopy moment maps

Let G denote a Lie group and $\mathfrak{g}$ its Lie algebra. Let $\Phi \colon G \times \mathcal{M} \to \mathcal{M}$ denote a left-action of G on the n-plectic manifold $(\mathcal{M}, \omega)$. That is, G acts on $\alpha \in \Omega^\bullet(\mathcal{M})$ from the left through the inverse pullback:

$$g \cdot \alpha \mapsto \Phi^*_{g^{-1}} \alpha \,, \tag{6.20}$$

where $\Phi_g \colon \mathcal{M} \to \mathcal{M}$ denotes the diffeomorphism that corresponds to g through the action ϕ. The corresponding infinitesimal action of the Lie algebra $\mathfrak{g}$ is denoted by the map:

$$v_- : \mathfrak{g} \to \mathfrak{X}(\mathcal{M}) \,, \quad x \mapsto v_x \,, \tag{6.21}$$

where:

$$v_x|_p = \frac{d}{dt} \exp(-tx) \cdot p|_{t=0} \qquad \forall p \in \mathcal{M} \,. \tag{6.22}$$

The vector field v_- is usually denoted in the literature as the *fundamental vector field* associated to the G-action Φ on $\mathcal{M}$.

In the context of symplectic geometry, we can equivalently write a moment map $\mu \colon \mathcal{M} \to \mathfrak{g}^*$ as a *comoment map*, namely a Lie algebra morphism $\mu^* \colon \mathfrak{g} \to C^\infty(\mathcal{M})$; see section 3.2 for more details. We will introduce in this section, closely following the seminal paper [18], the natural analog in multisymplectic geometry of the comoment map used in symplectic geometry. It is the so-called *map*.

Remark 6.1.11. Let us recall the definition of moment and comoment map. Let $(\mathcal{M}, \omega)$ be a symplectic manifold equipped with a G-action Φ. A moment map $\mu \colon \mathcal{M} \to \mathfrak{g}^*$ is a G-equivariant $\mathfrak{g}^*$-valued smooth function on $\mathcal{M}$. The moment map μ has to satisfy a particular condition respecto to ω. This notion can be also expressed as a comoment map, which is nothing but a Lie algebra homomorphism $\mu : \mathfrak{g} \to (C^\infty(\mathcal{M}), \{,\})$ from the Lie algebra $\mathfrak{g}$ of G, to the Poisson algebra on $C^\infty(\mathcal{M})$ that can be associated to the symplectic form.

Definition 6.1.12. Let G be a Lie group with Lie algebra $\mathfrak{g}$. Let $(\mathcal{M},\omega)$ be an n-plectic manifold which is equipped with a G-action Φ preserving ω and such that the corresponding $\mathfrak{g}$-action $x \mapsto v_x$ is through Hamiltonian vector fields. A *homotopy moment map* is a L_∞-algebra morphism $\mathfrak{f}\colon \mathfrak{g} \to L(\mathcal{M},\omega)$ making commutative the following diagram:

$$\begin{tikzcd} & L(\mathcal{M},\omega) \arrow[d, "\pi"] \\ \mathfrak{g} \arrow[ur, "\mathfrak{f}"] \arrow[r, "v_-"] & \mathfrak{X}_{\mathrm{Ham}}(\mathcal{M}) \end{tikzcd}$$

Hence, $\mathfrak{f}$ is a lift of $v_- : \mathfrak{g} \to \mathfrak{X}_{\mathrm{Ham}}(\mathcal{M})$ in the category of L_∞-algebras. This lift corresponds to an L_∞-morphism:

$$(f_k)_{k\geq 1} : \mathfrak{g} \to L_\infty(\mathcal{M},\omega)\ , \tag{6.23}$$

that in addition is required to satisfy:

$$-\iota_{v_y}\omega = d(f_1(y)) \qquad \text{for all } y \in \mathfrak{g}\,. \tag{6.24}$$

Notice that the condition $-\iota_{v_y}\omega = d(f_1(y))$ implies that v_y is the unique Hamiltonian vector field for $f_1(y) \in \Omega^{n-1}_{\mathrm{Ham}}(\mathcal{M})$. In addition, using proposition 5.2.6, we can rewrite the conditions on the components $(f_k)_{k\geq 1} : \mathfrak{g}^{\otimes k} \to L(\mathcal{M},\omega)$ of the L_∞-morphism as follows:

$$\sum_{1\leq i<j\leq k} (-1)^{i+j+1} f_{k-1}([y_i,y_j], y_1,\dots,\widehat{y_i},\dots,\widehat{y_j},\dots,y_k) \tag{6.25}$$

$$= df_k((y_1,\dots,y_k) + \xi(k)\iota(v_1\wedge\cdots\wedge v_k)\omega\,, \tag{6.26}$$

for $2 \leq k \leq n$ plus

$$\sum_{1\leq i<j\leq n+1} (-1)^{i+j+1} f_n([y_i,y_j], y_1,\dots,\widehat{y_i},\dots,\widehat{y_j},\dots,y_{n+1}) = \xi(n+1)\iota(v_1\wedge\cdots\wedge v_{n+1})\omega\,. \tag{6.27}$$

Here v_i stands for the vector field associated to y_i via the $\mathfrak{g}$-action. Please notice that the theorem 6.1.8 implies in particular that $L_\infty(\mathcal{M},\omega)$ satisfies 5.34.

Notice also that proposition 6.1.7 implies in particular that $v_{[x,y]} = [v_x, v_y]$ is a Hamiltonian vector field for

$$\{f_1(x), f_1(y)\} = l_2(f_1(x), f_1(y))\ . \tag{6.28}$$

In general, the map $f_1 : \mathfrak{g} \to \Omega^{n-1}_{\mathrm{Ham}}(\mathcal{M})$ will not preserve the bracket on $\mathfrak{g}$, *i.e.*, we will have

$$f_1([x,y]) \neq \{f_1(x), f_1(y)\}\ . \tag{6.29}$$

This is nice property should be expected, since the Lie bracket of $\mathfrak{g}$ satisfies the Jacobi identity but $\{\cdot,\cdot\}$ does not.

Definition 6.1.13. Let $(\mathcal{M}, \omega)$ be an n-plectic manifold. The action of a Lie group G on $(\mathcal{M}, \omega)$ is said to be Hamiltonian if an homotopy moment map for such action exists.

6.2 Multisymplectic diffeomorphisms and n-algebra morphisms

In this section we are going to study the relation between strict morphisms of L_∞-algebras and multisymplectic diffeomorphisms of the corresponding multisimplectic manifolds. That is, we want to know under which conditions, if any, we can conclude that a strict morphism of Lie n-algebras $\phi\colon L(M_1, \omega_1) \to L(M_2, \omega_2)$ *induces* a multisymplectic diffeomorphism between the corresponding n-plectic manifolds (M_a, ω_a), $a = 1, 2$. We know that in the symplectic case the answer is positive: two symplectic manifolds with corresponding isomorphic Poisson algebras are symplectomorphic. We will see that in the n-plectic case the answer is also positive, at least for a special class of n-plectic manifolds, those which are *locally homogeneous* with respect to the multisymplectic form. Hence, at least in those cases, the L_∞-algebra constructed on a multisymplectic manifold is powerful enough to contain important information about the manifold itself and its differential structure.

Let $\phi : L(M_2, \omega_2) \to L(M_1, \omega_1)$ be an strict Lie n-algebra morphism and let us write $\phi = (\phi_{1-n}, \ldots, \phi_0)$, where $\phi_i : L_i(M_2, \omega_2) \to L_i(M_1, \omega_1)\ ,\ i = 1-n, \ldots, 0$[2].

In order to relate strict Lie n-algebra morphisms and multisymplectic diffeomorphisms, we need first the existence of ϕ to imply the existence of a diffeomorphism $F : M_1 \to M_2$, which then must be checked to be a multisymplectic diffeomorphism. This is easily achieved by making use of the following lemma

Lemma 6.2.1. *Let $\mathcal{M}_a\, , a = 1, 2\,,$ be differentiable manifolds and $\psi : (C^\infty(\mathcal{M}_2), \cdot) \to (C^\infty(\mathcal{M}_1), \cdot)$ an algebra morphism, where $\cdot$ denotes the usual multiplication of functions. Then $\psi = F^*$, where $F : \mathcal{M}_1 \to \mathcal{M}_2$ is a smooth map. In addition, if ψ is an algebra isomorphism then F is a diffeomorphism.*

Proof. See theorem 4.2.36 in reference [76].

Hence, assuming that $\phi_{1-n} : C^\infty(\mathcal{M}_2) \to C^\infty(\mathcal{M}_1)$ is an algebra isomorphism from $\{C^\infty(\mathcal{M}_2), \cdot\}$ to $\{C^\infty(\mathcal{M}_1), \cdot\}$, we can conclude the existence of a diffeomorphism F from $\mathcal{M}_1$ to $\mathcal{M}_2$ such that:

$$\phi_{n-1} = F^*\,. \tag{6.30}$$

Since ϕ is an strict Lie n-algebra morphism, it preserves the maps $\{l^a_k\}_{k=1,\ldots,n+1}$ of the corresponding Lie n-algebra[3]:

$$d_2 \circ \phi = \phi \circ d_1\,, \tag{6.31}$$

$$\phi_{2-k} \circ l^2_k(\alpha_1, \cdots, \alpha_k) = l^1_k(\phi_0 \circ \alpha_1, \ldots, \phi_0 \circ \alpha_k)\ , \qquad \forall\ \alpha_1, \ldots, \alpha_k \in \Omega^{n-1}_{\mathrm{Ham}}(\mathcal{M}_2)\ , \qquad k = 2, \ldots, n+1\,. \tag{6.32}$$

Using now the definition (6.14) of the L_∞-algebra maps l_k, as well as (6.30), we obtain:

$$F^*\left\{\alpha^2_1, \cdots, \alpha^2_{n+1}\right\}_2 = \left\{\phi_0 \circ \alpha^2_1, \cdots, \phi_0 \circ \alpha^2_{n+1}\right\}_1\ , \qquad \forall\ \alpha^2_1, \ldots, \alpha^2_{n+1} \in \Omega^{n-1}_{\mathrm{Ham}}(\mathcal{M}_2)\,. \tag{6.33}$$

[2]For the precise definition of strict morphism of L_∞-algebras see 5.2.

[3]Notice that $l^a_1 = d_a$.

Assuming in addition that $\phi_i = F^*\,,\ i = 1-n, \dots, 0\,$, it can be proven that the initial set-up consisting of two n-plectic manifolds and a strict Lie-n algebra morphism ϕ is equivalent, in a precise sense to be specified in a moment, to considering a unique manifold $\mathcal{M}$ equipped with two n-plectic structures ω_1 and ω_2 such as $l^1_k = l^2_k\,,\ k = 1, \cdots, n+1$. Notice that the condition $\phi_0 = F^*$ is non-trivial, since for arbitrary diffeomorphisms we would have

$$F^* : \Omega^{n-1}_{\mathrm{Ham}}\left(\mathcal{M}_2\right) \to \Omega^{n-1}\left(\mathcal{M}_1\right)\,, \tag{6.34}$$

and we are requiring

$$F^* : \Omega^{n-1}_{\mathrm{Ham}}\left(\mathcal{M}_2\right) \to \Omega^{n-1}_{\mathrm{Ham}}\left(\mathcal{M}_1\right)\,. \tag{6.35}$$

We will assume then that the Lie-n algebra morphism is given by $\phi = F^*$, where $F : \mathcal{M}_1 \to \mathcal{M}_2$ is a diffeomorphism, and conclude then that F must be a multisymplectomorphism by studying the equivalent case of a unique manifold $\mathcal{M}_1$ equipped with two multisimplectic structures ω_1 and ω_2, such that the corresponding Lie-n algebras are equal. Let us first prove the equivalence of both cases.

Proposition 6.2.2. *Let* $\left(\mathcal{M}_a, \omega_a\right)\,,\ a = 1, 2\,,$ *be n-plectic manifolds,* $\left\{L\left(\mathcal{M}_a, \omega_a\right), l^a_k\right\}$ *denote the corresponding Lie n-algebras and* $\phi : \left\{L\left(\mathcal{M}_2, \omega_2\right), l^2_k\right\} \to \left\{L\left(\mathcal{M}_1, \omega_1\right), l^1_k\right\}$ *a strict Lie n-algebra morphism such that:*

$$\phi_i = F^* : L_i\left(M_2, \omega_2\right) \to L_i\left(M_1, \omega_1\right)\,,\ i = 1-n, \dots, 0\,, \tag{6.36}$$

where $F : \mathcal{M}_1 \to \mathcal{M}_2$ *is a diffeomorphism. Then, F is a multisymplectic diffeomorphism if and only if* $\left\{L\left(\mathcal{M}_1, \omega_1\right), l^1_k\right\} = \left\{L\left(\mathcal{M}_1, \tilde{\omega}_1 \equiv F^*\omega_2\right), \tilde{l}^1_k\right\}$ *implies* $\omega_1 = \tilde{\omega}_1$.

Proof. If $F : \mathcal{M}_1 \to \mathcal{M}_2$ is a multisymplectomorphism, then $F^*\omega_2 = \omega_1$ and therefore $\left\{L\left(\mathcal{M}_1, \omega_1\right), l^1_k\right\} = \left\{L\left(\mathcal{M}_1, \tilde{\omega}_1\right), \tilde{l}^1_k\right\}$ since $\tilde{\omega}_1 = \omega_1$, which in turn implies $\tilde{l}^1_k = l^1_k\,,\ k = 1, \cdots, n+1$.

On the other hand, if $\left\{L\left(\mathcal{M}_1, \omega_1\right), l^1_k\right\} = \left\{L\left(\mathcal{M}_1, \tilde{\omega}_1 \equiv F^*\omega_2\right), \tilde{l}^1_k\right\}$ implies $\omega_1 = \tilde{\omega}_1$., then $\omega_1 = F^*\omega_2$ and therefore F is a multisymplectomorphism. $\square$

In other words, proposition 6.2.2 simply states that the following diagram of strict-isomorphisms of L_∞-algebras commutes:

$$\begin{tikzcd}
& L\left(\mathcal{M}_1, F^*\omega_2\right) \arrow[dl, "\mathrm{Id}"'] \\
L\left(\mathcal{M}_1, \omega_1\right) & L\left(\mathcal{M}_2, \omega_2\right) \arrow[l, "\phi"'] \arrow[u, "F^*"']
\end{tikzcd}$$

Therefore, we will consider the equivalent situation of a unique manifold $\mathcal{M}$ equipped with two multisymplectic structures ω_1 and ω_2. Before proving the main result of this section, namely theorem 6.2.7, it is necessary to introduce the concept of *locally homogeneous manifold* and the lemma 6.2.5 of reference [2].

Definition 6.2.3. Let $\mathcal{M}$ be a differentiable manifold. Consider $p \in \mathcal{M}$ and a compact set $K \in \mathcal{M}$ such that $p \in K^{\circ}$[4]. A local Liouville or local Euler-like vector field at p, with respect to K, is a

[4] K° denotes the interior of K.

vector field Δ^p on $\mathcal{M}$ such that $\mathrm{supp}\Delta^p \equiv \overline{\{q\in\mathcal{M}|\Delta^p(q)\neq 0\}}\subset K$, and there exists a diffeomorphism $\phi:(\mathrm{supp}\Delta^p)^\circ\to\mathbb{R}^n$ such that $\phi_*\Delta^p=\Delta$, where $\Delta = x^i\frac{\partial}{\partial x^i}$ is the standard Liouville or dilation vector field in $\mathbb{R}^n$.

Definition 6.2.4. A differential form $\omega\in\Gamma\left(\Lambda^{(n+1)}\left(T^*\mathcal{M}\right)\right)$ is said to be *locally homogeneous* at $p\in\mathcal{M}$ if, for every open set U containing p, there exists a local Euler-like vector field Δ^p at p with respect to a compact set $K\subset U$ such that:

$$\mathcal{L}_{\Delta^p}\omega = f\omega\,,\qquad f\in C^\infty\left(\mathcal{M}\right)\,. \tag{6.37}$$

The form is said to be locally homogeneous if it is locally homogeneous for all $p\in\mathcal{M}$.

Obviously, out of $\mathrm{supp}\Delta^p$, the function f vanishes. A pair $(\mathcal{M},\omega)$ where ω is locally homogeneous n-plectic form is called a locally homogeneous n-plectic manifold. As examples of homogeneous n-plectic manifolds we can find symplectic manifolds and multicotangent bundles and in fact any oriented manifold equipped with its volume form.

The following lemmas[5] will play an important role in the proof of theorem 6.2.7.

Lemma 6.2.5. *Let $(\mathcal{M},\omega)$ be a locally homogeneous n-plectic manifold. Then, the family of hamiltonian vector fields span the tangent bundle of $\mathcal{M}$. That is*

$$T_p\mathcal{M} = \mathrm{span}\left\{v_p\,|\,v\in\Gamma\left(T\mathcal{M}\right)\,,\ i_v\omega = d\alpha_v\,,\ \alpha_v\in\Omega^{(n-1)}_{\mathrm{Ham}}(\mathcal{M})\right\}\,. \tag{6.38}$$

Proof. Let $(\mathcal{M},\omega)$ be a locally homogeneous n-plectic manifold, and let $v_p\in T_p\mathcal{M}$ be any vector at $p\in\mathcal{M}$. Let U be a contractible open neighborhood of p, which can be shrink in order to be contained in a coordinate chart $\mathcal{U}_\alpha$, with coordinates ϕ_α. In lemma 4.5 of reference [2], it was proven the existence of a vector field v_U on U, such that $\mathrm{supp}\,v_U\subset U$ is compact, $v_U|_p = v_p$ and

$$d\iota_{v_U}\omega = 0\,, \tag{6.39}$$

that is, $\iota_{v_U}\omega$ is closed. v_U can be extended trivially to all $\mathcal{M}$ by defining a vector field $v\in\Gamma\left(T\mathcal{M}\right)$ as follows:

$$v|_p \;=\; v_U|_p\,,\ p\in\mathrm{supp}\,v_U\,;\qquad v|_p = 0\,,\ p\notin\mathrm{supp}\,v_U\,, \tag{6.40}$$

We will prove now that v is a Hamiltonian vector field on $\mathcal{M}$. $\iota_{v_U}\omega$ is a closed n-form with compact support in U, that is $[\iota_{v_U}]\,\omega\in H^n_C\left(U\right)$. In addition, we can choose $U\subset\mathcal{U}_\alpha$ such that

$$U\simeq\mathbb{R}^d\,,\qquad d=\dim\mathcal{M}\,. \tag{6.41}$$

Therefore $H^n_C\left(U\right)\simeq H^n_C\left(\mathbb{R}^d\right)$ where $H^n_C\left(\mathbb{R}^d\right)$ denotes the n-th compactly supported cohomology group of $\mathbb{R}^d$. Noticing that $n<d$ we conclude that

$$H^n_C\left(U\right)\simeq H^n_C\left(\mathbb{R}^d\right)\simeq\{0\}\,. \tag{6.42}$$

Hence, there exist a $(n-1)$-form $\alpha_U\in\Omega^{n-1}\left(U\right)$ with compact support contained in U such that

$$\iota_{v_U}\omega = d\alpha_U\,. \tag{6.43}$$

[5]Lemma (6.2.5) is actually is a small generalization of lemma 4.5 in [2].

Now, extending α_U trivially to a $(n-1)$-form $\alpha \in \Omega^{n-1}(\mathcal{M})$ as follows

$$\alpha|_p = \alpha_U|_p\,,\ p \in \operatorname{supp}\alpha_U\,; \qquad \alpha|_p = 0\,,\ p \notin \operatorname{supp}\alpha_U\,, \tag{6.44}$$

we see that

$$\iota_v \omega = d\alpha\,. \tag{6.45}$$

Therefore, v is a Hamiltonian vector field in $\mathcal{M}$ that can be build as to give any vector $v|_p = v_p$ at $p \in \mathcal{M}$. We conclude then that $T_p\mathcal{M}$ is generated by Hamiltonian vector fields on $\mathcal{M}$ evaluated at $p \in \mathcal{M}$. □

Lemma 6.2.6. *Let $\mathcal{M}$ be a multisymplectic manifold equipped with two n-plectic structures ω_1 and ω_2 such that the corresponding Lie-n algebras are equal, namely $L(\mathcal{M},\omega_2) = L(\mathcal{M},\omega_1)$. Let us assume that at least one of the n-plectic structures, say ω_1, is locally homogeneous. Then*

$$v^1_\alpha = v^2_\alpha\,, \qquad \forall\ \alpha \in \Omega^{(n-1)}_{\mathrm{Ham}}(\mathcal{M})\,, \tag{6.46}$$

where v^a_α, $a = 1,2$ is the Hamiltonian vector field of α respect to ω_a, that is

$$d\alpha = \iota_{v^a_\alpha}\omega_a\,, \qquad a = 1,2\,. \tag{6.47}$$

Proof. Let $\alpha \in \Omega^{(n-1)}_{\mathrm{Ham}}(\mathcal{M})$ and let v^a_α the Hamiltonian vector of α respect to ω_a. Since by assumption $L(\mathcal{M},\omega_2) = L(\mathcal{M},\omega_1)$, we can write

$$l^1_2(\alpha,\beta) = l^2_2(\alpha,\beta)\,, \qquad \forall\ \alpha,\beta \in \Omega^{(n-1)}_{\mathrm{Ham}}(\mathcal{M})\,, \tag{6.48}$$

which, in turn, implies

$$\omega_1\left(v^1_\alpha, v^1_\beta, \dots\right) = \omega_2\left(v^2_\alpha, v^2_\beta, \dots\right)\,, \qquad \forall\ \alpha,\beta \in \Omega^{(n-1)}_{\mathrm{Ham}}(\mathcal{M})\,, \tag{6.49}$$

where v^a_α is the Hamiltonian vector field of α associated to ω_a. Using now that

$$d\alpha = -\iota_{v^1_\alpha}\omega_1 = -\iota_{v^2_\alpha}\omega_2\,, \qquad \forall\alpha \in \Omega^{(n-1)}_{\mathrm{Ham}}(\mathcal{M})\,, \tag{6.50}$$

we can rewrite equation (6.49) as follows

$$\omega_1\left(v^1_\alpha, v^1_\beta, \dots\right) = \omega_1\left(v^1_\alpha, v^2_\beta, \dots\right)\,, \qquad \forall\ \alpha,\beta \in \Omega^{(n-1)}_{\mathrm{Ham}}(\mathcal{M})\,, \tag{6.51}$$

The non-degeneracy of ω_1 together with lemma 6.2.5 finally implies

$$v^1_\beta = v^2_\beta\,, \qquad \forall\ \beta \in \Omega^{(n-1)}_{\mathrm{Ham}}(\mathcal{M})\,. \tag{6.52}$$

□

Theorem 6.2.7. *Let $\mathcal{M}$ be a differentiable manifold equiped with two multisimplectic structures ω_1 and ω_2, such that at least one of them is locally homogeneous. Then $L(\mathcal{M},\omega_2) = L(\mathcal{M},\omega_1)$ if and only if $\omega_1 = \omega_2$.*

Proof. If $\omega_1 = \omega_2$ it is obvious that $L(\mathcal{M},\omega_2) = L(\mathcal{M},\omega_1)$. On the other hand, let us assume that $L(\mathcal{M},\omega_2) = L(\mathcal{M},\omega_1)$. In particular, the underlying complex and the multilinear brackets constructed from ω_1 and ω_2 must be equal. We can write then

$$l^1_{(n+1)}\left(\alpha_1,\dots,\alpha_{(n+1)}\right) = l^2_{(n+1)}\left(\alpha_1,\dots,\alpha_{(n+1)}\right)\,, \qquad \forall\ \alpha_1,\dots,\alpha_{(n+1)} \in \Omega^{(n-1)}_{\mathrm{Ham}}(\mathcal{M})\,, \tag{6.53}$$

and therefore

$$i_{v_{\alpha_1}\wedge\cdots\wedge v_{\alpha_{(n+1)}}}\omega_1 = i_{v_{\alpha_1}\wedge\cdots\wedge v_{\alpha_{(n+1)}}}\omega_2\,, \qquad \forall\, v_{\alpha_1},\ldots,v_{\alpha_{(n+1)}} \in \mathfrak{X}_{\mathrm{Ham}}\left(\mathcal{M}\right)\,, \tag{6.54}$$

where we have used lemma 6.2.6 in order to use the same Hamiltonian vector fields for ω_1 and ω_2. Evaluating now 6.54 point-wise we obtain, using lemma 6.2.5

$$i_{v_1\wedge\cdots\wedge v_{(n+1)}}\omega_1|_p = i_{v_1\wedge\cdots\wedge v_{(n+1)}}\omega_2|_p\,, \qquad \forall\, v_1|_p,\ldots,v_{(n+1)}|_p \in T_p\left(\mathcal{M}\right)\,. \tag{6.55}$$

We conclude hence that $\omega_1|_p = \omega_2|_p$ for all $p \in \mathcal{M}$ and therefore $\omega_1 = \omega_2$. □

From theorem 6.2.7 and propostion 6.2.2 we immediately conclude the final result of this section, namely

Theorem 6.2.8. *Let* $\left(\mathcal{M}_a,\omega_a\right)$, $a = 1,2$, *be locally homogeneous multisymplectic manifolds, let* $\left\{L\left(M_a,\omega_a\right), l_k^a\right\}$ *denote the corresponding* L_∞ *algebras and let* $\phi : \left\{L\left(\mathcal{M}_2,\omega_2\right), l_k^2\right\} \to \left\{L\left(\mathcal{M}_1,\omega_1\right), l_k^1\right\}$ *an strict* L_∞*-isomorphism such that*

$$\phi_i = F^* : L_i\left(\mathcal{M}_2,\omega_2\right) \to L_i\left(\mathcal{M}_1,\omega_1\right)\,,\ \ i = 1-n,\ldots,0\,, \tag{6.56}$$

where $F : \mathcal{M}_1 \to \mathcal{M}_2$ *is a diffeomorphism. Then,* F *is also a multisymplectic diffeomorphism, that is,* $F^*\omega_2 = \omega_1$.

Proof. Direct consequence of proposition (6.2.2) and theorem (6.2.7). □

6.3 Product manifolds and Lie n-algebra morphisms

Consider two multisymplectic manifolds $\left(\mathcal{M}_a,\omega_a\right)$, $a = 1,2$, where ω_a is an n_a-plectic structure defined on $\mathcal{M}_a$. The goal of this section is, roughly speaking, to study the relation between the n_a-Lie algebra $L\left(\mathcal{M}_a,\omega_a\right)$ constructed over $\mathcal{M}_a$ and the Lie-n algebra $L\left(\mathcal{M},\omega\right)$ constructed over the product manifold, $\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2$, equipped with the $n = (n_1 + n_2 + 1)$-plectic structure $\omega = \mathrm{pr}_1^*\omega_1 \wedge \mathrm{pr}_2^*\omega_2$. Here $\mathrm{pr}_a : \mathcal{M} \to \mathcal{M}_a$ denotes the canonical projection.

Finding such a relation is relevant for at least two reasons. First, it help us to understand how n-plectic Lie algebras are related to the corresponding multisymplectic manifolds in a deeper way, since it give us information about how they behave when some operation is performed in the manifold, in this case the cartesian product. Secondly, it is relevant in order to construct an homotopy moment map[6] for the product manifold, assuming the homotopy moment maps for $\left(\mathcal{M}_1,\omega_1\right)$ and $\left(\mathcal{M}_2,\omega_2\right)$ exist. Indeed, let G_a be a Lie group with Lie algebra $\mathfrak{g}_a$. Let $\left(\mathcal{M}_a,\omega_a\right)$ be a n_a-plectic manifold equipped with a G_a action which preserves ω_a and such that the infinitesimal $\mathfrak{g}_a$ action is via Hamiltonian fields. Let us assume that the corresponding homotopy moment maps exist and are given by $f_a : \mathfrak{g}_a \to L\left(\mathcal{M}_a,\omega_a\right)$. In that case, if $\mathrm{H} : L\left(\mathcal{M}_1,\omega_1\right) \oplus L\left(\mathcal{M}_2,\omega_2\right) \to L\left(\mathcal{M},\omega\right)$ is a L_∞-morphism, then the composition of $f_1 \oplus f_2$ and H is a very reasonable homotopy moment map candidate for the product manifold $G_1 \times G_2 \circlearrowleft \left(\mathcal{M},\omega\right)$. To obtain H by brute force seems to be a very involved task to perform in general. It will turn out to be easier to directly construct an L_∞-morphism F from $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ to $\left(\mathcal{M}_1 \times \mathcal{M}_2,\omega\right)$, making use of f_a, which can also be used to make an educated guess for H, as it is illustrated by the following diagram:

[6] See section 6.1.1.

$$\begin{tikzcd}
L\left(\mathcal{M}_1,\omega_1\right)\oplus L\left(\mathcal{M}_2,\omega_2\right) \arrow[r, "\mathrm{H}"] & L\left(\mathcal{M},\omega\right) \\
\mathfrak{g}_1\oplus\mathfrak{g}_2 \arrow[u, "f_1", "f_2"'] \arrow[ur, "F"'] &
\end{tikzcd}$$

Since it will be useful in the following, let us remember that the tangent bundle of the product manifold $\mathcal{M}$ can be written as follows[7]

$$T\mathcal{M} = \mathrm{pr}_1^* T\mathcal{M}_1 \oplus \mathrm{pr}_2^* T\mathcal{M}_2 \,, \tag{6.57}$$

and therefore

$$\Gamma\left(T\mathcal{M}\right) = \Gamma\left(\mathrm{pr}_1^* T\mathcal{M}_1\right) \oplus \Gamma\left(\mathrm{pr}_2^* T\mathcal{M}_2\right) \,. \tag{6.58}$$

In particular, it holds $\mathrm{pr}_a^*\Gamma\left(T\mathcal{M}_a\right) \subset \Gamma\left(\mathrm{pr}_a^* T\mathcal{M}_a\right) \subset \Gamma\left(T\mathcal{M}\right)$. The first task is now to check that ω is indeed an n-plectic structure on $\mathcal{M}$, provided ω_a is an n_a-plectic structure on $\mathcal{M}_a$. It is straightforward to see that ω is closed

$$d\omega = \mathrm{pr}_1^* d\omega_1 \wedge \mathrm{pr}_2\omega_2 + (-1)^{n_1}\mathrm{pr}_1^*\omega_1 \wedge \mathrm{pr}_2^* d\omega_2 = 0 \,. \tag{6.59}$$

To see that it is non-degenerate, let us assume that there exists a vector field $v \in \Gamma\left(\mathcal{M}\right)$ such that $i_v\omega = 0$. There exist then sections $X_a \in \Gamma\left(\mathrm{pr}_a^* T\mathcal{M}_a\right)$ such that:

$$v = X_1 \oplus X_2 \,, \tag{6.60}$$

and therefore:

$$\iota_v\omega = \iota_v \mathrm{pr}_1^*\omega_1 \wedge \mathrm{pr}_2^*\omega_2 + (-1)^{n_1+1}\mathrm{pr}_1^*\omega_1 \wedge \iota_v \mathrm{pr}_2^*\omega_2 \,, \tag{6.61}$$

implies $v_a = 0$ and $v = 0$ since ω_a is non-degenerate. Let us consider now $X_{\alpha_a} \in \mathfrak{X}_{\mathrm{Ham}}\left(\mathcal{M}_a\right) = \Gamma_{\mathrm{Ham}}\left(T\mathcal{M}_a\right)$ and construct the vector field

$$X_\alpha = \mathrm{pr}_1^* X_{\alpha_1} + \mathrm{pr}_2^* X_{\alpha_2} \,. \tag{6.62}$$

We have then

$$i_{X_\alpha}\omega = -d\left[\mathrm{pr}_1^*\alpha_1 \wedge \mathrm{pr}_2^*\omega + \mathrm{pr}_1^*\omega_1 \wedge \mathrm{pr}_2^*\alpha_2\right] = -d\alpha \,, \tag{6.63}$$

and hence X_α is a hamiltonian vector field for ω with hamiltonian (n_1+n_2)-form α, which is of course defined up to a closed form. Therefore we have

$$\mathrm{pr}_1^*\Gamma_{\mathrm{Ham}}\left(T\mathcal{M}_1\right) + \mathrm{pr}_2^*\Gamma_{\mathrm{Ham}}\left(T\mathcal{M}_1\right) \subseteq \Gamma_{\mathrm{Ham}}\left(T\mathcal{M}\right) \,. \tag{6.64}$$

The following example shows that in general

$$\mathrm{pr}_1^*\Gamma_{\mathrm{Ham}}\left(T\mathcal{M}_1\right) + \mathrm{pr}_2^*\Gamma_{\mathrm{Ham}}\left(T\mathcal{M}_1\right) \neq \Gamma_{\mathrm{Ham}}\left(T\mathcal{M}\right) \,. \tag{6.65}$$

[7]See section 2.2.

Example 6.3.1. Let $\mathcal{M}_a = \mathbb{R}^2$ with coordinates (x_a^1, x_a^2) equipped with the volume form $\omega_a = f_a\, dx_a^1 \wedge dx_a^2$, where $f_a \in C^\infty(\mathbb{R}^2)$ is a no-where vanishing, non-constant, differentiable function. Then

$$\mathcal{M} = \mathbb{R}^4\,, \qquad \omega = f_1 f_2\, dx^1 \wedge \cdots \wedge dx^4 \tag{6.66}$$

where the coordinates of $\mathbb{R}^4$ are denoted by $(x^1, \ldots, x^4)$. Then

$$X_\alpha = -\frac{1}{f_1 f_2}\frac{\partial}{\partial x^1}\,, \qquad \alpha = x^2 dx^3 \wedge dx^4\,, \tag{6.67}$$

is a Hamiltonian vector field which cannot be written as a sum of Hamiltonian vector fields X_{α_a} of $(\mathbb{R}^2, \omega_a)$.

We are going to define now two applications h_Ω and $h_{\mathfrak{X}}$ as follows

$$\begin{aligned} h_\Omega : \Omega^{n_1-1}_{\mathrm{Ham}}(\mathcal{M}_1) \oplus \Omega^{n_2-1}_{\mathrm{Ham}}(\mathcal{M}_2) &\to \Omega^{n_1+n_2}_{\mathrm{Ham}}(\mathcal{M}) \\ \alpha_1 \oplus \alpha_2 &\mapsto \mathrm{pr}_1^*\alpha_1 \wedge \mathrm{pr}_2^*\omega_2 + \mathrm{pr}_1^*\omega_1 \wedge \mathrm{pr}_2^*\alpha_2\,, \end{aligned} \tag{6.68}$$

$$\begin{aligned} h_{\mathfrak{X}} : \mathfrak{X}_{\mathrm{Ham}}(\mathcal{M}_1) \oplus \mathfrak{X}_{\mathrm{Ham}}(\mathcal{M}_2) &\to \mathfrak{X}_{\mathrm{Ham}}(\mathcal{M}) \\ X_{\alpha_1} \oplus X_{\alpha_2} &\mapsto \mathrm{pr}_1^* X_{\alpha_1} + \mathrm{pr}_2^* X_{\alpha_2}\,, \end{aligned} \tag{6.69}$$

which make the following diagram commutative

$$\begin{tikzcd} \Omega^{n_1-1}_{\mathrm{Ham}}(\mathcal{M}_1) \oplus \Omega^{n_2-1}_{\mathrm{Ham}}(\mathcal{M}_2) \arrow[r, "h_\Omega"] \arrow[d, "j"'] & \Omega^{n_1+n_2}_{\mathrm{Ham}}(\mathcal{M}) \arrow[d, "k"] \\ \mathfrak{X}_{\mathrm{Ham}}(\mathcal{M}_1) \oplus \mathfrak{X}_{\mathrm{Ham}}(\mathcal{M}_2) \arrow[r, "h_{\mathfrak{X}}"] & \mathfrak{X}_{\mathrm{Ham}}(\mathcal{M}) \end{tikzcd}$$

Here $j(\alpha_1 \oplus \alpha_2) = X_{\alpha_1} \oplus X_{\alpha_2}$ and $k(\alpha) = X_\alpha$. The vector space $\mathfrak{X}_{\mathrm{Ham}}(\mathcal{M}_1) \oplus \mathfrak{X}_{\mathrm{Ham}}(\mathcal{M}_2)$ over the real numbers $\mathbb{R}$ can be endowed with an $\mathbb{R}$-linear Lie bracket

$$[\cdot,\cdot]_0 : \mathfrak{X}_{\mathrm{Ham}}(\mathcal{M}_1) \oplus \mathfrak{X}_{\mathrm{Ham}}(\mathcal{M}_2) \oplus \mathfrak{X}_{\mathrm{Ham}}(\mathcal{M}_1) \oplus \mathfrak{X}_{\mathrm{Ham}}(\mathcal{M}_2) \to \mathfrak{X}_{\mathrm{Ham}}(\mathcal{M}_1) \oplus \mathfrak{X}_{\mathrm{Ham}}(\mathcal{M}_2)\,, \tag{6.70}$$

defined as

$$[X_{\alpha_1} \oplus X_{\alpha_2}, X_{\beta_1} \oplus X_{\beta_2}]_0 = [X_{\alpha_1}, X_{\beta_1}]_1 \oplus [X_{\alpha_2}, X_{\beta_2}]_2\,, \tag{6.71}$$

where $[\cdot,\cdot]_a : \mathfrak{X}_{\mathrm{Ham}}(\mathcal{M}_a) \oplus \mathfrak{X}_{\mathrm{Ham}}(\mathcal{M}_a) \to \mathfrak{X}_{\mathrm{Ham}}(\mathcal{M}_a)$ is the canonical Lie brackets defined on $\mathfrak{X}(\mathcal{M}_a)$ and evaluated on $\mathfrak{X}_{\mathrm{Ham}}(\mathcal{M}_a)$. It can be easily seen that h_{X} preserves the Lie bracket, that is

$$h_{\mathfrak{X}}\left([X_{\alpha_1} \oplus X_{\alpha_2}, X_{\beta_1} \oplus X_{\beta_2}]_0\right) = [h_{\mathfrak{X}}(X_{\alpha_1} \oplus X_{\beta_1}), h_{\mathfrak{X}}(X_{\alpha_2} \oplus X_{\beta_2})]\,. \tag{6.72}$$

Similarly, $\Omega^{n_1-1}_{\mathrm{Ham}}(\mathcal{M}_1) \oplus \Omega^{n_2-1}_{\mathrm{Ham}}(\mathcal{M}_2)$ can be endowed with a bracket

$$\{\cdot,\cdot\}_0 : \Omega^{n_1-1}_{\mathrm{Ham}}(\mathcal{M}_1) \oplus \Omega^{n_2-1}_{\mathrm{Ham}}(\mathcal{M}_2) \oplus \Omega^{n_1-1}_{\mathrm{Ham}}(\mathcal{M}_1) \oplus \Omega^{n_2-1}_{\mathrm{Ham}}(\mathcal{M}_2) \to \Omega^{n_1-1}_{\mathrm{Ham}}(\mathcal{M}_1) \oplus \Omega^{n_2-1}_{\mathrm{Ham}}(\mathcal{M}_2) \quad (6.73)$$

defined as follows

$$\{\alpha_1 \oplus \alpha_2, \beta_1 \oplus \beta_2\}_0 = \{\alpha_1, \beta_1\}_1 \oplus \{\alpha_2, \beta_2\}_2 \,, \quad (6.74)$$

where $\{\cdot,\cdot\}_a : \Omega^{n_a-1}_{\mathrm{Ham}}(\mathcal{M}_a) \oplus \Omega^{n_a-1}_{\mathrm{Ham}}(\mathcal{M}_a) \to \Omega^{n_a-1}_{\mathrm{Ham}}(\mathcal{M}_a)$ is the canonical Hamiltonian bracket defined on $\Omega^{n_a-1}_{\mathrm{Ham}}(\mathcal{M}_a)$. However, in this case, h_Ω does not preserve the bracket $\{\cdot,\cdot\}_0$, namely

$$h_\Omega(\{\alpha_1 \oplus \alpha_2, \beta_1 \oplus \beta_2\}_0) = \{h_\Omega(\alpha_1 \oplus \alpha_2), h_\Omega(\beta_1 \oplus \beta_2)\} + (-1)^{n_1} d\left[\mathrm{pr}_1^*\alpha_1 \wedge \mathrm{pr}_2^* d\beta_2 - \mathrm{pr}_1^*\beta_1 \wedge \mathrm{pr}_2^* d\alpha_2\right] . \quad (6.75)$$

As we have previously stated, the goal of this section is to relate the n_a-Lie algebras $L(\mathcal{M}_a, \omega_a)$ constructued over $(\mathcal{M}_a, \omega_a)$ to the $(n_1 + n_2 + 1)$-Lie algebra $L(\mathcal{M}, \omega)$ constructed over $(\mathcal{M}, \omega)$. More precisely, we want to construct an Lie-n algebra morphism from $L(\mathcal{M}_1, \omega_1) \oplus L(\mathcal{M}_2, \omega_2)$ to $L(\mathcal{M}, \omega)$. To construct such morphism, h_Ω is going to be extremely relevant. The idea is to use h_Ω as the very first component of the L_∞-algebra morphism H. This suggested by the fact that h_Ω fails to preserve the bracket $\{\cdot,\cdot\}_0$ by an exact form, something that is characteristic of the corresponding component in a L_∞-morphism. Assuming therefore that $H_1 = h_\Omega$, we expect to obtain the form af all the other components of H by imposing the defining and consistency conditions that H has to obey, namely (6.25) and (6.27). However, this is an extremely involved procedure, so we have been able to check it only in the simplest case, where $(\mathcal{M}_a, \omega_a)$ are both symplectic spaces. In any case, let us stress that we expect the procedure to hold in full generality.

6.3.1 $(\mathcal{M}_a, \omega_a)$ Symplectic manifolds

Since $(\mathcal{M}_a, \omega_a)$ is a 1-plectic manifold, we have that $(\mathcal{M}, \omega)$ is a 3-plectic manifold. Consequently, the cochain complex L of the Lie 3-algebra $L(\mathcal{M}_a, \omega_a)$ is given by

$$L : C^\infty(\mathcal{M}) \to \Omega^1(\mathcal{M}) \to \Omega^2(\mathcal{M}) \to \Omega^3_{\mathrm{Ham}}(\mathcal{M}) \,, \quad (6.76)$$

where the coboundary operator is the usual de Rham exterior derivative. On the other hand, the cochain complex L_a which underlies the 1-Lie algebra $L(\mathcal{M}_a, \omega_a)$ is simply

$$L_a : C^\infty(\mathcal{M}_a) \,. \quad (6.77)$$

In this simpler situation, $L(\mathcal{M}_1, \omega_1) \oplus L(\mathcal{M}_2, \omega_2)$ is just a regular Lie-algebra, so we can apply propostion 5.2.6 in order to obtain the remaining components of the L_∞-morphism H from the Lie algebra $L(\mathcal{M}_1, \omega_1) \oplus L(\mathcal{M}_2, \omega_2)$ to the 3-Lie algebra $L(\mathcal{M}, \omega)$. H consists of three maps

$$H_k : [C^\infty(\mathcal{M}_1) \times C^\infty(\mathcal{M}_2)]^{\otimes k} \to L \,, \qquad k = 1, 2, 3 \,. \quad (6.78)$$

Please notice that we are imposing $H_1 = h_\Omega$. The two remaining components will be find by imposing the defining conditions of a L_∞ morphism on H, which in proposition 5.2.6 have been adapted and simplified to the case of a Lie algebra as the domain of the morphism. As expected, the procedure is consistent and H_2 and H_3 can be determined, making H into an honest L_∞-morphism. They are given by[8]

[8] In equation (6.80) we have omitted the pr_a^* in order to make more readable the expression.

$$\mathrm{H}_2\left(f_1, f_2, g_1, g_2\right) = \frac{1}{2}\left(\mathrm{pr}_1^* f_1 \wedge \mathrm{pr}_2^* dg_2 - \mathrm{pr}_1^* df_1 \wedge \mathrm{pr}_2^* g_2 - \mathrm{pr}_1^* g_1 \wedge \mathrm{pr}_2^* df_2 + \mathrm{pr}_1^* dg_1 \wedge \mathrm{pr}_2^* f_2\right) , \tag{6.79}$$

$$\mathrm{H}_3\left(f_1, f_2, g_1, g_2, h_1, h_2\right) = \frac{1}{2}\left(f_1\left\{g_2, h_2\right\} + f_2\left\{g_1, h_1\right\} - g_1\left\{f_2, h_2\right\} - g_2\left\{f_1, h_1\right\} + h_1\left\{f_2, g_2\right\} + h_2\left\{f_1, g_1\right\}\right) , \tag{6.80}$$

for all $f_a, g_a, h_a \in C^\infty\left(\mathcal{M}_a\right)\ ,\ a = 1, 2$. As explained in section 6.1.1, in order for H to be an homotopy moment map we have to check that

$$-\iota_{v_x}\omega = d\left(f_1(x)\right) , \tag{6.81}$$

which holds by equation (6.63). Therefore, we have constructed explicitly a Lie-n algebra morphism from $L\left(\mathcal{M}_1, \omega_1\right) \times L\left(\mathcal{M}_2, \omega_2\right)$ to $L\left(\mathcal{M}, \omega\right)$, and we expect the same procedure to hold in the general case. However, it turns out to be too involved to be carried out explicitly and a different approach is needed.

6.4 Product homotopy moment maps

Since finding H explicitly seems to be a too complicated task to be performed by brute force, in this section we are going to pursue a different goal. We are going to build an L_∞-morphism from[9] $\mathfrak{g}_a \oplus \mathfrak{g}_b$ to $L\left(\mathcal{M} = \mathcal{M}_a \times \mathcal{M}_b, \omega = \mathrm{pr}_a^*\omega_a \wedge \mathrm{pr}_b^*\omega_b\right)$, assuming that there exist homotopy moment maps f_C for $G_C \circlearrowright \left(\mathcal{M}_C, \omega_C\right)\ ,\ C = a, b$. Here $\mathfrak{g}_C$ is the Lie algebras of the Lie group G_C. The L_∞-morphism F will give us an homotopy moment map for the product manifold in terms of homotopy moment maps of the factors. Besides, F may give us also the opportunity to make an educated guess for H : $L\left(\mathcal{M}_a, \omega_a\right) \oplus L\left(\mathcal{M}_b, \omega_b\right) \to L\left(\mathcal{M}, \omega\right)$.

Hence, let $\left(M_C, \omega_C\right), C = a, b$, be a n-plectic manifold and let G_C be a Lie group, with Lie algebra $\mathfrak{g}_C$, which acts on $\left(M_C, \omega_C\right)$ in a Hamiltonian way, with corresponding homotopy moment map $f^C : \mathfrak{g}_C \to L_\infty\left(M_C, \omega_C\right)$. Then $G \equiv G_a \times G_b$ acts on the $(n_a + n_b + 1)$-plectic manifold[10]

$$\left(M \equiv M_a \times M_b\ ,\ \omega \equiv \omega_a \wedge \omega_b\right) .$$

The main theorem of this section is Theorem 6.4.3, where from the above data we explicitly construct a homotopy moment map $F : \mathfrak{g}_a \otimes \mathfrak{g}_b \to L_\infty(M, \omega)$.

6.4.1 The construction of F

We first recall a few facts from [18] [20]. Let (M, ω) be a n-plectic manifold, and G a Lie group acting on M preserving ω. The manifold M and the Lie algebra $\mathfrak{g}$ give rise to a double complex:

$$K := (\wedge^{\geq 1}\mathfrak{g}^* \otimes \Omega(M), d_\mathfrak{g}, d) ,$$

where $d_\mathfrak{g}$ is the Chevallier-Eilenberg differential of $\mathfrak{g}$ and d is the de Rham differential of M. We consider the total complex with differential:

$$d_{tot} := d_\mathfrak{g} \otimes 1 + 1 \otimes d .$$

[9]In this section we are going to use a different notation, more suitable for the expressions that we will find. a and b are not indices but labels that denote different objects.

[10]We will slightly abuse the notation, denoting a differential form on M_C and its pullback to $M_a \times M_b$, via the canonical projection, by the same symbol.

Hence, on an element of $\wedge^k\mathfrak{g}^*\otimes\Omega(M)$, d_{tot} acts as $d_\mathfrak{g}+(-1)^k d$.

For any G-invariant $\sigma\in\Omega^N(M)$ define:

$$\sigma^k\colon \wedge^k\mathfrak{g}\to\Omega^{N-k}(M),\ \ (x_1,\dots,x_k)\mapsto\iota(v_1\wedge\cdots\wedge v_k)\sigma\,,$$

and:

$$\tilde{\sigma}:=\sum_{k=1}^{N}(-1)^{k-1}\sigma^k\,. \tag{6.82}$$

Since each σ^k can be viewed as an element of $\wedge^k\mathfrak{g}^*\otimes\Omega^{N-k}(M)$, it follows that σ can be viewed as an element of K of total degree N. It turns out that $\tilde{\omega}$ is d_{tot}-closed, as a consequence of the fact that ω is a closed form. The link to homotopy moment maps is given by [18, Prop. 2.5], which we reproduce for the reader's convenience:

Proposition 6.4.1. *Let $\varphi=\varphi_1+\cdots+\varphi_n$, with $\varphi_k\in\wedge^k\mathfrak{g}^*\otimes\Omega^{n-k}(M)$. Then: $d_{tot}\varphi=\tilde{\omega}$ iff*

$$f_k:=\varsigma(k)\varphi_k\colon\ \wedge^k\mathfrak{g}\to\Omega^{n-k}(M),$$

for $k=1,\dots,n$, are the components of a homotopy moment map for the action of G on (M,ω).

Now we apply the previous machinery to the manifolds $M_a, M_b, M_a\times M_b$ and the data given at the beginning of this section. For each of these three manifolds we obtain a double complex, which we will denote by (K_a,d^a_{tot}), (K_b,d^b_{tot}) and (K,d_{tot}) respectively.

Lemma 6.4.2. *Let $\varphi^C\in K^C$ be of degree n_C. If $d^C_{tot}\varphi^C=\widetilde{\omega_C}$ for $C=a,b$, then $d_{tot}\varphi=\widetilde{\omega_a\wedge\omega_b}$ where:*

$$\varphi=\frac{1}{2}(-\varphi^a\widetilde{\omega_b}+(-1)^{n_a}\widetilde{\omega_a}\varphi^b)+(\varphi^a\omega_b+(-1)^{n_a+1}\omega_a\varphi^b)\in K\,.$$

Proof. First notice that:

$$\widetilde{\omega_a\wedge\omega_b}=-\widetilde{\omega_a}\widetilde{\omega_b}+\widetilde{\omega_a}\omega_b+\omega_a\widetilde{\omega_b}\,. \tag{6.83}$$

This is a consequence of: $\widehat{\omega_a}\widehat{\omega_b}=\widehat{\omega_a\wedge\omega_b}$ for $\widehat{\omega_C}:=\omega_C-\widetilde{\omega_C}$.

Now we exhibit d_{tot}-primitives for each of the three summands in eq. (6.83).

$$\begin{aligned} d_{tot}(\varphi^a\widetilde{\omega_b}+(-1)^{n_a+1}\widetilde{\omega_a}\varphi^b) &= d^a_{tot}\varphi^a\widetilde{\omega_b}+(-1)^{n_a}\varphi^a d^b_{tot}\widetilde{\omega_b}+(-1)^{n_a+1}d^a_{tot}\widetilde{\omega_a}\varphi^b+\widetilde{\omega_a}d^b_{tot}\varphi^b\\ &=2\widetilde{\omega_a}\widetilde{\omega_b}\,,\end{aligned}$$

where in the last equation we used our assumption and $d^C_{tot}\widetilde{\omega_C}=0$, which holds by [18].

Further

$$d_{tot}(\varphi^a\omega_b)=d^a_{tot}\varphi^a\omega_b+(-1)^{n_a}\varphi^a d^b_{tot}\omega_b=\widetilde{\omega_a}\omega_b,$$

where in the last equation to compute $d^b_{tot}\omega_b=0$ we have to enlarge the double complex K^b to include $\wedge^0(\mathfrak{g}_b)^*\otimes\Omega(M_b)\cong\Omega(M_b)$.

Similarly,

$$d_{tot}((-1)^{n_a+1}\omega_a\varphi^b)=\omega_a\widetilde{\omega_b}.$$

□

Applying Prop. 6.4.1, the d_{tot}-primitive of $\widetilde{\omega_a\wedge\omega_b}$ obtained in Lemma 6.4.2 allows us to construct a homotopy moment map for the $\mathfrak{g}$ action on $(M,\omega_a\wedge\omega_b)$:

Theorem 6.4.3. *Let G_C be a Lie group with Lie algebra $\mathfrak{g}_C$, where $C = a, b$. Let (M_C, ω_C) be a n_C-plectic manifold equipped with a G_C action admitting a homotopy moment map $f^C : \mathfrak{g}_C \to L_\infty(M_C, \omega_C)$. Then the action of $G_a \times G_b$ on $(M, \omega) := (M_a \times M_b, \omega_a \wedge \omega_b)$ admits a homotopy moment map with components determined by graded skew-symmetry and the formulae ($k = 1, \ldots, n_1+n_2+1$)*

$$\begin{aligned} F_k : (\mathfrak{g}_a \oplus \mathfrak{g}_b)^{\otimes k} &\to L_\infty(M, \omega) \\ \left(x_a^1, \ldots, x_a^m, x_b^1, \ldots, x_b^l\right) &\mapsto c^a_{m,l}\, f^a_m\left(x_a^1, \ldots, x_a^m\right) \wedge \iota_{1,\ldots,l}\omega_b \\ &+ c^b_{m,l}\, \iota_{1,\ldots,m}\omega_a \wedge f^b_l\left(x_b^1, \ldots, x_b^l\right), \end{aligned} \tag{6.84}$$

where $m, l \geq 0$ with $m + l = k$, $x_a^i \in \mathfrak{g}_a$ and $x_b^i \in \mathfrak{g}_b$. Here we define $f^a_0 = f^b_0 = 0$ and:

$$\iota_{1,\ldots,i}\,\omega_C = \iota\left(v_{f_1^C(x_C^1)} \wedge \cdots \wedge v_{f_1^C(x_C^i)}\right)\omega_C .$$

The coefficients are defined as follows for all $m \geq 1, l \geq 1$:

$$c^a_{m,l} = \frac{1}{2}\varsigma(m+l)\varsigma(m)(-1)^{(n_a+1-m)l}, \tag{6.85}$$

$$c^b_{m,l} = \frac{1}{2}\varsigma(m+l)\varsigma(l)(-1)^{(n_a+1-m)(l+1)}, \tag{6.86}$$

and:

$$c^a_{m,0} = 1, \qquad c^b_{0,l} = (-1)^{(l+1)(n_a+1)}.$$

Recall that $\varsigma(k) = -(-1)^{\frac{k(k+1)}{2}}$.

Remark 6.4.4. The formula for F_k simplifies once written using the operations $[\ldots]$ introduced in Def. 6.1.10:

$$\begin{aligned} F_k(x_a^1, \ldots, x_a^m, x_b^1, \ldots, x_b^l) = \widehat{c^a_{m,l}}\, f^a_m\left(x_a^1, \ldots, x_a^m\right) \wedge \left[f_1^b(x_b^1), \ldots, f_1^b(x_b^l)\right] \\ + \widehat{c^b_{m,l}}\left[f_1^a(x_a^1), \ldots, f_1^a(x_a^m)\right] \wedge f^b_l\left(x_b^1, \ldots, x_b^l\right), \end{aligned}$$

where for all $m \geq 1, l \geq 1$:

$$\widehat{c^a_{m,l}} = -\frac{1}{2}(-1)^{(n_a+1)l}, \tag{6.87}$$

$$\widehat{c^b_{m,l}} = -\frac{1}{2}(-1)^{(n_a+1)(l+1)+m}, \tag{6.88}$$

and:

$$\widehat{c^a_{m,0}} = -1, \qquad \widehat{c^b_{0,l}} = -(-1)^{(l+1)(n_a+1)}.$$

This is a straightforward consequence of $\varsigma(m)\varsigma(l)\varsigma(m+l) = -(-1)^{ml}$ for all integers $m, l \geq 0$.

Proof. Prop. 6.4.1 and Lemma 6.4.2 deliver a homotopy moment map $F\colon \mathfrak{g}_a \oplus \mathfrak{g}_b \to L_\infty(M_a \times M_b, \omega)$ whose components F_k, for $k = 1, \dots, n_a + n_b + 1$, are given by:

$$F_k = \varsigma(k)\varphi_k\,,$$

where:

$$\varphi = \frac{1}{2}(-\varphi^a\widetilde{\omega_b} + (-1)^{n_a}\widetilde{\omega_a}\varphi^b) + (\varphi^a\omega_b + (-1)^{n_a+1}\omega_a\varphi^b)\,. \tag{6.89}$$

Let us point out that:

$$\varphi_k \in \Lambda^k\left(\mathfrak{g}_a^* \oplus \mathfrak{g}_b^*\right) \otimes \Omega^{(n_a+n_b+1-k)}\left(M_a \times M_b\right)\,.$$

In order to prove the theorem we just have to write F_k using equation (6.89) and $f_k^a = \varsigma(k)\varphi_k^a$, $f_k^b = \varsigma(k)\varphi_k^b$. We do so evaluating the components of F on elements of $\mathfrak{g}_a$ and of $\mathfrak{g}_b$.

We have:

$$\begin{aligned} F_m(x_a^1, \dots, x_a^m) = \varsigma(m)\varphi_m(x_a^1, \dots, x_a^m) &= \varsigma(m)\varphi_m^a(x_a^1, \dots, x_a^m) \wedge \omega_b \\ &= f_m^a(x_1^a, \dots, x_m^a)\, \wedge \omega_b\,, \end{aligned}$$

using that $\varphi_m^a = \varsigma(m) f_m^a$ in the last equality. In the second equality we used eq. (6.89) (notice that on the r.h.s. of eq. (6.89), only the summand $\varphi^a\omega_b$ gives a contribution). We conclude that:

$$c_{m,0}^a = 1\,, \qquad m \geq 1\,.$$

Let us take now:

$$\begin{aligned} F_l(x_b^1, \dots, x_b^l) = \varsigma(l)\varphi_l(x_b^1, \dots, x_b^l) &= \varsigma(l)(-1)^{n_a+1}(\omega_a\varphi_l^b)(x_b^1, \dots, x_b^l) \\ &= (-1)^{n_a+1}(\omega_a f_l^b)(x_b^1, \dots, x_b^l) \\ &= (-1)^{(n_a+1)(l+1)}\omega_a \wedge f_l^b(x_b^1, \dots, x_b^l)\,. \end{aligned}$$

The last equality holds since[11], if we pick a basis $\{\xi_i^b\}$ of $\mathfrak{g}_b^*$ and write f_l^b as a sum of terms of the form $\xi_{i_1}^b \wedge \dots \wedge \xi_{i_l}^b \otimes \beta \in \Lambda^l\left(\mathfrak{g}_b^*\right) \otimes \Omega^{(n_b-l)}\left(M_b\right)$, then:

$$(1 \otimes \omega_a)(\xi_{i_1}^b \wedge \dots \wedge \xi_{i_l}^b \otimes \beta) = (-1)^{(n_a+1)l}\xi_{i_1}^b \wedge \dots \wedge \xi_{i_l}^b \otimes (\omega_a \wedge \beta)\,.$$

We obtain:

$$c_{0,l}^b = (-1)^{(n_a+1)(l+1)}\,, \qquad l \geq 1\,.$$

For $m, l \geq 1$ consider:

$$\begin{aligned} &F_{m+l}(x_a^1, \dots, x_a^m, x_b^1, \dots, x_b^l) \\ =&\varsigma(m+l)\varphi_{m+l}(x_a^1, \dots, x_a^m, x_b^1, \dots, x_b^l) \\ =&\varsigma(m+l)\frac{1}{2}\left(-\varphi_m^a(\widetilde{\omega_b})_l + (-1)^{n_a}(\widetilde{\omega_a})_m\varphi_l^b\right)(x_a^1, \dots, x_a^m, x_b^1, \dots, x_b^l) \\ =&\varsigma(m+l)\frac{1}{2}\left(-\varsigma(m)(-1)^{l-1}f_m^a{\omega_b}^l + (-1)^{n_a}(-1)^{m-1}\varsigma(l){\omega_a}^m f_l^b\right)(x_a^1, \dots, x_a^m, x_b^1, \dots, x_b^l)\,, \end{aligned}$$

[11] We are slightly abusing the notation by denoting the product of two elements in the double-complexes K_C or K and the wedge product of forms simply by juxtaposition.

where in the last equality we used eq. (6.82). We have:

$$-\varphi^a_m \widetilde{\omega_b}_l = (f^a_m {\omega_b}^l)(x^1_a, \ldots, x^m_a, x^1_b, \ldots, x^l_b) = (-1)^{(n_a - m)l} f^a_m(x^1_a, \ldots, x^m_a) \wedge {\omega_b}^l(x^1_b, \ldots, x^l_b) \,,$$

using $f^a_m \in \Lambda \mathfrak{g}^*_a \otimes \Omega^{n_a - m}(M_a)$ and ${\omega_b}^l \in \Lambda^l \mathfrak{g}^*_b \otimes \Omega(M_b)$. Therefore:

$$c^a_{m,l} = \frac{1}{2} \varsigma(l+m) \varsigma(m) (-1)^{(n_a+1-m)l} \,.$$

Similarly,

$$-\varphi^a_m \widetilde{\omega_b}_l = ({\omega_a}^m f^b_l)(x^1_a, \ldots, x^m_a, x^1_b, \ldots, x^l_b) = (-1)^{(n_a+1-m)l} {\omega_a}^m(x^1_a, \ldots, x^m_a) \wedge f^b_l(x^1_b, \ldots, x^l_b) \,,$$

using ${\omega_a}^m \in \Lambda \mathfrak{g}^*_a \otimes \Omega^{n_a+1-m}(M_a)$ and $f^b_l \in \Lambda^l \mathfrak{g}^*_b \otimes \Omega(M_b)$. Hence:

$$c^b_{m,l} = \frac{1}{2} \varsigma(l+m) \varsigma(l) (-1)^{(n_a+1-m)(l+1)} \,.$$

□

Example 6.4.5. We spell out the homotopy moment map constructed in theorem 6.4.3 in the case that M_a and M_b are symplectic manifolds, i.e. $n_a = n_b = 1$. In that case $f^a \colon \mathfrak{g}_a \to C^\infty(M_a)$ is an ordinary comoment map, just like f^b, and (M, ω) is a 3-plectic manifold. One obtains:

$$\begin{aligned}
F_1(x_a \oplus x_b) &= f^a(x_a) \cdot \omega_b + \omega_a \cdot f^b(x_b) \\
F_2(x_a \oplus x_b, y_a \oplus y_b) &= \frac{1}{2} \left(-f^a(x_a) \cdot \iota_{v_{f^b(y_b)}} \omega_b + \iota_{v_{f^a(x_a)}} \omega_a \cdot f^b(y_b) \right) - (x \leftrightarrow y) \\
F_3(x_a \oplus x_b, y_a \oplus y_b, z_a \oplus z_b) &= -\frac{1}{2} \left(f^a(x_a) \cdot \iota_{v_{f^b(y_b)} \wedge v_{f^b(z_b)}} \omega_b + \iota_{v_{f^a(x_a)} \wedge v_{f^a(y_a)}} \omega_a \cdot f^b(z_b) \right) + c.p.
\end{aligned}$$

where $x_C, y_C, z_C \in \mathfrak{g}_C$ for $C = a, b$ and "c.p." denotes cyclic permutations of x, y, z.

6.4.2 Non-associativity of the construction

The construction of homotopy moment maps for product manifolds given in theorem 6.4.3 is not associative.

More precisely: for $C = a, b, c$ let G_C be a Lie group with Lie algebra $\mathfrak{g}_C$, acting on a n_C-plectic manifold (M_C, ω_C) with homotopy moment map $f^C : \mathfrak{g}_C \to L_\infty(M_C, \omega_C)$. Denote by $f^a * f^b$ the homotopy moment map for the action of $G_a \times G_b$ on $(M_a \times M_b, \omega_a \wedge \omega_b)$ constructed in theorem 6.4.3. Then

$$(f^a * f^b) * f^c \neq f^a * (f^b * f^c) \,, \tag{6.90}$$

as one can see from a straightforward computation using the fact that $c^a_{m,l} = \pm\frac{1}{2}$ for $m \geq 1, l \geq 1$.

Indeed, the construction of the d_{tot}-primitives done in lemma 6.4.2 is also not associative: denote by φ^C the elements of K^C corresponding to the homotopy moment maps f^C (via proposition 6.4.1). If we denote by $\varphi^a * \varphi^b$ the d_{tot}-primitive of $\widetilde{\omega_a \wedge \omega_b}$ constructed in Lemma 6.4.2, then $(\varphi^a * \varphi^b) * \varphi^c$ and $\varphi^a * (\varphi^b * \varphi^c)$ are different[12] primitives for $(\widetilde{(\omega_a \wedge \omega_b)} \wedge \omega_c) = (\omega_a \wedge \widetilde{(\omega_b \wedge \omega_c)})$. The difference between these two primitives is:

[12]One could hope that redefining $\varphi^a * \varphi^b$ by adding a real multiple of $d_{tot}(\varphi^a \varphi^b)$ to it might remove this issue, but this is not the case.

$$\frac{1}{4}(-\varphi^a\widetilde{\omega_b}\widetilde{\omega_c}+\widetilde{\omega_a}\widetilde{\omega_b}\varphi^c)=d_{tot}\left(-\frac{1}{4}\varphi^a\widetilde{\omega_b}\varphi^c\right).$$

Hence the two homotopy moment maps appearing in eq. (6.90) are *inner equivalent* in the sense of [18, Remark 7.10]. This notion of inner equivalence is the one that arises naturally considering the complex $\wedge^{\geq 1}(\mathfrak{g}_a\times\mathfrak{g}_b\times\mathfrak{g}_c)^*\otimes\Omega(M_a\times M_b\times M_c)$, and can be characterized as equivalence of L_∞-morphisms (see [18, Prop. A2]).

Under quite restrictive conditions, there is another way to construct homotopy moment maps for product manifolds, which does have the property of being associative in the sense above.

Remark 6.4.6. Given an action of G_a on the n_a-plectic manifold (M_a,ω_a), the theorem [18, Theorem 6.8] provides a map:

$$\Phi_{M_a}\colon \{\text{Closed extensions of } \omega_a \text{ in } C_{G_a}(M_a)\}\to\{\text{Homotopy moment maps for } (M_a,\omega_a)\},$$

where $C_{G_a}(M_a)=(S\mathfrak{g}_a^*\otimes\Omega(M_a))^{G_a}$ is the Cartan model for the equivariant cohomology of the G_a action on M_a (it is a differential graded algebra).

This map is not surjective in general [18]. It is also not injective in general: by the formulae in [18, Thm. 6.8] it is clear that, if $\mathfrak{g}_a$ is a abelian Lie algebra, then the component lying in $(S^2\mathfrak{g}_a^*\otimes\Omega^{n_a-3}(M_a))^{G_a}$ of a closed extension ψ^a can not be recovered from the homotopy moment map $\Phi_{M_a}(\psi^a)$.

However, in the cases in which Φ_{M_a} and Φ_{M_b} are injective[13], one can carry out the following construction:

if homotopy moment maps f^C for (M_C,ω_C) arising from closed extensions in the Cartan model $(C=a,b)$ are given, then

$$\Phi_{M_a\times M_b}(\psi^a\cdot\psi^b) \tag{6.91}$$

is a homotopy moment map for $(M_a\times M_b,\omega_a\wedge\omega_b)$, where ψ^C is determined by $\Phi_{M_C}(\psi^C)=f^C$, and the dot denotes the product in the Cartan model $C_{G_a\times G_b}(M_a\times M_b)$. This prescription has the property of being associative, in the sense above, for the simple reason that the algebra structure in the Cartan model is associative.

In the special case of symplectic manifolds (M_a,ω_a) and (M_b,ω_b), the injectivity assumption is satisfied. The above prescription (6.91) delivers a homotopy moment map H for $(M_a\times M_b,\omega_a\wedge\omega_b)$, which as expected is different from the one F obtained in Ex. 6.4.5: we have $H_1=F_1$, $H_2=F_2$, but

$$\begin{aligned}H_3(x_a\oplus x_b,y_a\oplus y_b,z_a\oplus z_b)=&\frac{2}{3}F_3(x_a\oplus x_b,y_a\oplus y_b,z_a\oplus z_b)\\&-\frac{1}{6}\left(f^a(x_a)f^b([y_b,z_b])+f^a([y_a,z_a])f^b(x_b)+c.p.\right)\end{aligned}$$

where $x_C,y_C,z_C\in\mathfrak{g}_C$ for $C=a,b$.

6.5 Application: homotopy moment maps for iterated powers (M,ω^m)

In Section 6.4 we have shown how to build a homotopy moment map for the product manifold of two multisymplectic manifolds, assuming that a homotopy moment map for the individual manifolds exist. Here we apply this construction to some specific examples of geometrical interest: powers of closed forms and Hyperkähler manifolds.

[13] The same prescription does not seem to work without the injectivity assumption, for in that case it seems to depend on the choice of ψ^a and ψ^b. In view of the formulae in [18, Thm. 6.8], the technical reason behind this is the following: if $P_2^a\in S^2\mathfrak{g}_a^*$ is a quadratic polynomial on the Lie algebra $\mathfrak{g}_a$, then the total skew-symmetrization of $P_2^a([\cdot,\cdot],[\cdot,\cdot])\colon\mathfrak{g}_a^{\otimes 4}\to\mathbb{R}$ does not seem to be determined by the total skew-symmetrization of $P_2^a(\cdot,[\cdot,\cdot])\colon\mathfrak{g}_a^{\otimes 3}\to\mathbb{R}$.

6.5.1 Restrictions

Let G be a Lie group with Lie algebra $\mathfrak{g}$, acting on a n-plectic manifold (M,ω) with homotopy moment map $f\colon \mathfrak{g} \to L_\infty(M,\omega)$. One obtains new actions, either restricting to a Lie subgroup of G or to an invariant submanifold of (M,ω). We display homotopy moment maps for both cases.

Lemma 6.5.1. *Let $H \subset G$ be a Lie subgroup, and denote by $j\colon \mathfrak{h} \hookrightarrow \mathfrak{g}$ the inclusion of its Lie algebra. The restricted action of H on (M,ω) has homotopy moment map $f \circ j\colon \mathfrak{h} \to L_\infty(M,\omega)$.*

Proof. The Lie algebra morphism j is in particular an L_∞-morphism, so $f \circ j$ also is. Since eq. (6.24) holds for all $x \in \mathfrak{g}$, in particular it holds for all $x \in \mathfrak{h}$. □

Lemma 6.5.2. *Let $N \overset{i}{\hookrightarrow} M$ a G-invariant submanifold of M. Then the action $G \circlearrowleft (N, i^*\omega)$ is Hamiltonian with homotopy moment map $i^* \circ f : \mathfrak{g} \to L_\infty(N, i^*\omega)$.*

Proof. According to definition 6.1.12, we have to show that

$$f^N := i^* \circ f : \mathfrak{g} \to L_\infty(N, i^*\omega)$$

is an L_∞-morphism such that

$$-\iota_{(v_x)^N} i^*\omega = df_1^N(x)\,, \qquad \forall\, x \in \mathfrak{g}, \tag{6.92}$$

where $(v_x)^N$, which is a generator of the action on N, denotes the restriction of the vector field v_x to N.

Eq. (6.92) follows simply by applying the pullback i^* to Eq. (6.24). To show that f^N is an L_∞-morphism, let us introduce the following L_∞-subalgebra of $L_\infty(M,\omega)$:

$$L^N(M,\omega) = C^\infty(M) \oplus \Omega^1(M) \oplus \cdots \oplus \widetilde{\Omega}^{n-1}_{\mathrm{Ham}}(M)\ ,$$

where:

$$\widetilde{\Omega}^{n-1}_{\mathrm{Ham}}(M) = \left\{ \alpha \in \Omega^{n-1}_{\mathrm{Ham}}(M) : \exists \text{ a Hamiltonian vector field of } \alpha \text{ tangent to } N \right\}\ .$$

Since $L_\infty(M,\omega)$ and $L^N(M,\omega)$ are equal in every component except for the degree zero component, in order to see that $L^N(M,\omega)$ is really a L_∞-subalgebra of $L_\infty(M,\omega)$, we only have to check that the binary bracket l_2 of $L_\infty(M,\omega)$ restricts to $\widetilde{\Omega}^{n-1}_{\mathrm{Ham}}(M)$. This is indeed the case since given any two Hamiltonian forms α and β and respective Hamiltonian vector fields v_α, v_β, a Hamiltonian vector field for $l_2(\alpha,\beta)$ is given by the Lie bracket $[v_\alpha, v_\beta]$, which of course is tangent to N whenever both v_α and v_β are.

Notice that the homotopy moment map $f\colon \mathfrak{g} \to L_\infty(M,\omega)$ takes values in $L^N_\infty(M,\omega)$, that is,

$$f_k(x) \in L^N(M,\omega)\ , \qquad \forall\, x \in \mathfrak{g}^{\otimes k} \qquad k \geq 1\,. \tag{6.93}$$

To prove this, since $L_\infty(M,\omega)$ and $L^N(M,\omega)$ are equal in every component but the zero one, we have to check equation (6.93) only in the $k=1$ case, that is, we have to prove that

$$f_1(x) \in \widetilde{\Omega}^{n-1}_{\mathrm{Ham}}(M)\ , \qquad \forall\, x \in \mathfrak{g}\,.$$

It holds since a Hamiltonian vector field of $f_1(x)$ is the generator of the action v_x, which is tangent to N by assumption.

Next, notice that the pullback of forms

$$i^*\colon L^N_\infty(M,\omega) \to L_\infty(N, i^*\omega)$$

is[14] a (strict) L_∞-morphism, as a consequence of the facts that i^* commutes with the de Rham differential and due to the definition of $\widetilde{\Omega}^{n-1}_{\mathrm{Ham}}(M)$. We conclude that $i^* \circ f : \mathfrak{g} \to L_\infty(N, i^*\omega)$ is a homotopy moment map. □

6.5.2 Actions on $(M, \omega \wedge \omega)$

Let us consider two multisymplectic manifolds (M_C, ω_C), $C = a, b$. We assume that there is a Hamiltonian action of a Lie group $G_C \circlearrowleft M_C$ with corresponding homotopy moment map $f^C : \mathfrak{g}_C \to L_\infty(M_C, \omega_C)$. By theorem 6.4.3 we know that there is also a Hamiltonian action:

$$G_a \times G_b \circlearrowleft (M_a \times M_b,\, \omega_a \wedge \omega_b)\ , \tag{6.94}$$

with homotopy moment map F given by theorem 6.4.3.

Assume now that $G_a = G_b =: G$, whose Lie algebra we denote by $\mathfrak{g}$. One can restrict the action (6.94) to the diagonal $\Delta G = \{(g, g) : g \in G\}$ of $G \times G$:

$$\Delta G \circlearrowleft (M_a \times M_b,\, \omega_a \wedge \omega_b)\,. \tag{6.95}$$

By Lemma 6.5.1, a homotopy moment map for this action is:

$$F \circ j : \Delta\mathfrak{g} \to L_\infty(M_a \times M_b,\, \omega_a \wedge \omega_b)\ ,$$

where

$$j\colon \Delta\mathfrak{g} = \{(x, x) : x \in \mathfrak{g}\} \to \mathfrak{g} \oplus \mathfrak{g} \tag{6.96}$$

is the inclusion. By the isomorphism $G \simeq \Delta G, g \mapsto (g, g)$ we can view eq. (6.95) as an action of the Lie group G, and j as a map $\mathfrak{g} \simeq \Delta\mathfrak{g} \to \mathfrak{g} \oplus \mathfrak{g}$.

Now we specialize even further, taking $M_a = M_b =: M$, $\omega_a = \omega_b =: \omega$ and $f^a = f^b$.
The diagonal ΔM of $M \times M$ is invariant under the action of ΔG. Therefore, using the inclusion

$$i : \Delta M \hookrightarrow M \times M \tag{6.97}$$

and the identification $M \simeq \Delta M$ we obtain by restriction an action of G on M:

$$G \simeq \Delta G \circlearrowleft (\Delta M, i^*(\omega \wedge \omega)) \simeq (M, \omega \wedge \omega)\,.$$

Of course, this is interesting only when ω has even degree, for otherwise $\omega \wedge \omega = 0$. Lemma 6.5.2 states that this action is Hamiltonian with homotopy moment map given by

$$i^* F \circ j : \mathfrak{g} \to L_\infty(M, \omega \wedge \omega)\ ,$$

where F is as in theorem 6.4.3.

Remark 6.5.3. If an action $G \circlearrowleft (M, \omega)$ is Hamiltonian, then the action $G \circlearrowleft (M, \omega^m)$, $m \in \mathbb{N}$, is also Hamiltonian. This follows from a slight variation of the above reasoning, allowing ω_a and ω_b to be different.

Remark 6.5.4. The above reasoning also leads to the following more general statement. Consider again, for $C = a, b$, actions $G_C \circlearrowleft M_C$ with corresponding homotopy moment maps f^C. Assume now that there is a manifold B and G_C-equivariant submersions $\pi_C\colon M_C \to B$. Then the diagonal action of G on the

[14] However the map $L_\infty(M, \omega) \to L_\infty(N, i^*\omega)$ given by pullback of forms is not an L_∞-morphism. This is the reason we need to introduce $L^N_\infty(M, \omega)$.

fiber product $M_a \times_B M_b = (\pi_a \times \pi_b)^{-1}(\Delta B)$, endowed with the pullback by the inclusion of $\omega_a \wedge \omega_b$, admits a homotopy moment map.

The special case $M_a = M_b = B$ with $\pi_a = \pi_b = Id$ delivers $(M, \omega \wedge \omega)$. Another interesting special case arises when $\pi_C \colon M_C \to B$ are principal G_C-bundles (in that case the action on B is trivial).

Making more explicit the formula for $i^*F \circ j$, we obtain:

Proposition 6.5.5. *Let G be a Lie group with Lie algebra $\mathfrak{g}$, and fix an action of G on an n-plectic manifold (M,ω) with homotopy moment map $f \colon \mathfrak{g} \to L_\infty(M,\omega)$, where n is odd. Then the G action on $(M, \omega \wedge \omega)$ has a homotopy moment map, with components ($k = 1, \ldots, 2n+1$)*

$$\begin{aligned} \mathfrak{g}^{\otimes k} &\to L_\infty(M, \omega \wedge \omega) \\ x^1 \otimes \cdots \otimes x^k &\mapsto 2 \sum_{m=1}^{k} \sum_{\sigma \in Sh_{m,k-m}} (-1)^\sigma c^a_{m,k-m} f_m \left(x^{\sigma(1)}, \ldots, x^{\sigma(m)}\right) \wedge \iota_{\sigma(m+1),\ldots,\sigma(k)}\omega. \end{aligned} \tag{6.98}$$

Remark 6.5.6. The above double sum consist of $2^k - 1$ summands.

Proof. Fix $k \geq 1$ and $x^1 \wedge \cdots \wedge x^k \in \wedge^k \mathfrak{g}$. Notice that:

$$j(x^1) \wedge \cdots \wedge j(x^k) \in \wedge^k(\mathfrak{g} \oplus \mathfrak{g})$$

is the sum of 2^k monomials in a natural way. For instance, introducing the notation $j(x) = x_a \oplus x_b$, one has $j(x^1) \wedge j(x^2) = x^1_a \wedge x^2_a + x^1_a \wedge x^2_b + x^1_b \wedge x^2_a + x^1_b \wedge x^2_b$. Let X denote one of these monomials, let m be the number of elements in X decorated by the index "a", and $l := k - m$.

If $m = 0$ or $l = 0$, it is clear by Thm. 6.4.3 that $(i^*(F_k))(X) = F(X) \wedge \omega$.

Hence we consider only the case that $m, l \neq 0$. X can be written as

$$(-1)^\sigma x_a^{\sigma(1)} \wedge \cdots \wedge x_a^{\sigma(m)} \wedge x_b^{\sigma(m+1)} \wedge \cdots \wedge x_b^{\sigma(k)}$$

for a unique $\sigma \in Sh_{m,l}$. By theorem 6.4.3 we have:

$$\begin{aligned} F_k(X) = (-1)^\sigma \Big[& c^a_{m,l}\, f_m \left(x_a^{\sigma(1)}, \ldots, x_a^{\sigma(m)}\right) \wedge \iota_{\sigma(m+1),\ldots,\sigma(k)}\omega \\ & + c^b_{m,l}\, \iota_{\sigma(1),\ldots,\sigma(m)}\omega \wedge f_l \left(x_b^{\sigma(m+1)}, \ldots, x_b^{\sigma(k)}\right) \Big]. \end{aligned} \tag{6.99}$$

Denote by Y the monomial obtained from X interchanging each index "a" with the index "b". Notice that Y can be written as

$$(-1)^\tau x_a^{\tau(1)} \wedge \cdots \wedge x_a^{\tau(l)} \wedge x_b^{\tau(l+1)} \wedge \cdots \wedge x_b^{\tau(k)}$$

for a unique $\tau \in Sh(l,m)$. One can check that the first summand of $F_k(X)$ in eq. (6.99) agrees exactly with the second summand of $F_k(Y)$. Hence:

$$\begin{aligned} (i^*(F_k))(X+Y) = 2\Big[& (-1)^\sigma c^a_{m,l}\, f_m \left(x^{\sigma(1)}, \ldots, x^{\sigma(m)}\right) \wedge \iota_{\sigma(m+1),\ldots,\sigma(k)}\omega \\ & + \ (-1)^\tau c^a_{l,m}\, f_m \left(x^{\tau(1)}, \ldots, x^{\tau(l)}\right) \wedge \iota_{\tau(l+1),\ldots,\tau(k)}\omega \Big]. \end{aligned}$$

Pairing two by two as above all the summands of $m(x^1) \wedge \cdots \wedge m(x^k)$ and summing up, we see that $(i^*(F_k) \circ j)(x^1 \wedge \cdots \wedge x^k)$ equals the expression given in the statement of this proposition. □

Not all the homotopy moment maps for $(M, \omega \wedge \omega)$ arise from homotopy moment maps for (M, ω) as in Prop. 6.5.5, as the following example shows.

Example 6.5.7. Consider the symplectic manifold $M := S^1 \times S^1 \times S^1 \times \mathbb{R}$ with canonical "coordinates" $\theta_1, \theta_2, \theta_3, x_4$, and symplectic form $\omega = d\theta_1 \wedge d\theta_2 + d\theta_3 \wedge dx_4$. The action of the circle on M with generator $\frac{\partial}{\partial \theta_1}$ is by symplectomorphisms, but does not admit a moment map since $d\theta_2$ is not exact.

On the other hand $\omega \wedge \omega = 2d\theta_1 \wedge d\theta_2 \wedge d\theta_3 \wedge dx_4$ is exact with invariant primitive (for instance, as primitive take $-2x_4 d\theta_1 \wedge d\theta_2 \wedge d\theta_3$). Therefore by [18] there is a homotopy moment maps for $\omega \wedge \omega$, constructed canonically using this primitive.

6.5.3 Hyperkähler manifolds

The results in this subsection are closely related to Martin Callies' results in [77].

Definition 6.5.8. A **Hyperkähler manifold** is a Riemannian manifold (M, g) equipped with three complex structures $J_i : TM \to TM\,, i = 1, 2, 3\,,$ which satisfy the quaternionic relations $J_i^2 = J_1 J_2 J_3 = -1$ and are covariantly constant with respect to the Levi-Civita connection ∇ associated to g, that is, $\nabla J_i = 0\,, i = 1, 2, 3\,.$ We say then that (g, J_1, J_2, J_3) is a Hyperkähler structure on M.

As a consequence of the definition of Hyperkähler manifold, M is also equipped with three symplectic two-forms $\omega_i\,, i = 1, 2, 3\,,$ as follows

$$\omega_i\,(u, v) = g\,(J_i u, v)\;, \quad u, v \in \mathfrak{X}(M)\,, \quad i = 1, 2, 3\,.$$

Remark 6.5.9. Notice that ω_i is non-degenerate as a consequence of g and J being non-degenerate and it is closed as a consequence of J_i being covariantly constant. In fact, we have $\nabla \omega_i = 0$ for $i = 1, 2, 3$. If $a_i \in \mathbb{R}\,, i = 1, 2, 3\,,$ with $\sum_{i=1}^3 a_i^2 = 1$, then $\sum_{i=1}^3 a_i J_i$ is a complex structure un M , and g is Kähler respect to it, with Kähler form $\sum_{i=1}^3 a_i \omega_i$. Hence, a Hyperkähler manifold M is equipped with a *sphere* of complex structures and Kähler forms.

A Hyperkähler manifold can be also characterized as a $4k$-dimensional (real) Riemannian manifold with Riemannian holonomy contained in $Sp(k)$, where $k \geq 1$. Since $Sp(k) \subset SU(2k)$, every Hyperkähler manifold is Calabi-Yau and Ricci-flat. Notice that the natural representation of $Sp(k)$ on $\mathbb{R}^{4k}$ preserves three complex structures $J_i\,, i = 1, 2, 3\,,$ that satisfy the quaternionic relations $J_i^2 = J_1 J_2 J_3 = -1$.

It turns out that

$$\Omega := \sum_{i=1}^{3} \omega_i \wedge \omega_i$$

is a 3-plectic form.

The following Lemma follows immediately from definition 6.1.12 using equations (6.25) and (6.27) (or alternatively from proposition 6.4.1).

Lemma 6.5.10. *Suppose we are given an action of a Lie group H on a manifold N preserving n-plectic forms Ω_1 and Ω_2, with homotopy moment maps F^1 and F^2 respectively. Then the action of H on $(N, \Omega_1 + \Omega_2)$ has homotopy moment map $F^1 + F^2$.*

Proposition 6.5.11. *Let G be a Lie group acting on the Hyperkähler manifold M. Assume that (M, ω_i) admits an equivariant moment map f^i, for $i = 1, 2, 3$. Then the G action on the 3-plectic manifold (M, Ω) admits a homotopy moment map, constructed canonically out of f^1, f^2, f^3.*

Proof. Since f^i is a moment map for ω_i, Prop. 6.5.5 provides a homotopy moment map F^i for $\omega_i \wedge \omega_i$, for $i = 1, 2, 3$. A homotopy moment map for Ω is then given by $F^1 + F^2 + F^3$, by Lemma 6.5.10. □

Not all homotopy moment maps for Ω arise from moment maps for the ω_i, as the following variation of Ex. 6.5.7 shows.

Example 6.5.12. Consider the Hyperkähler manifold $\mathbb{R}^4$ with the canonical metric and the complex structures J_1, J_2, J_3 given by quaternionic multiplication by $i, j, k \in \mathbb{H} = \mathbb{R}^4$. Dividing by the lattice $\mathbb{Z}^3 \times \{0\}$ we obtain a Hyperkähler structure on $M := S^1 \times S^1 \times S^1 \times \mathbb{R}$ (the product of the 3-torus with the real line), on which we have induced "coordinates" $\theta_1, \theta_2, \theta_3, x_4$. The symplectic structures on M associated to the distinguished complex structures are:

$$\omega_1 = d\theta_1 \wedge d\theta_2 + d\theta_3 \wedge dx_4, \quad \omega_2 = d\theta_1 \wedge d\theta_3 - d\theta_2 \wedge dx_4, \quad \omega_3 = d\theta_1 \wedge dx_4 + d\theta_2 \wedge d\theta_3.$$

The action of the circle on M with generator $\frac{\partial}{\partial\theta_1}$ preserves each ω_i, however ω_1 and ω_2 have no moment map for this action. On the other hand, it is easily computed that $\Omega := \sum_{i=1}^{3} \omega_i \wedge \omega_i = 6 d\theta_1 \wedge d\theta_2 \wedge d\theta_3 \wedge dx_4$, and Ω admits a homotopy moment map as we explained in Ex. 6.5.7.

6.6 Embeddings of L_∞-algebras associated to closed differential forms

Let (M_C, ω_C) be a n_C-plectic manifold, $C = a\,, b$. We consider the $n_a + n_b + 1$-plectic manifold

$$(M \equiv M_a \times M_b, \omega \equiv \omega_a \wedge \omega_b)\ .$$

Being (M_C, ω_C) a n_C-plectic manifold, it is equipped with a Lie n_C- algebra $L_\infty(M_C, \omega_C)$, constructed exclusively out of ω_C and the de Rahm differential d. The purpose of this section is to find an L_∞-morphism

$$H\colon L_\infty(M_a, \omega_a) \oplus L_\infty(M_b, \omega_b) \rightsquigarrow L_\infty(M_a \times M_b, \omega_a \wedge \omega_b) \tag{6.100}$$

whose first component is an embedding. We will exhibit such a morphism in Thm. 6.6.2.

Remark 6.6.1. As in the previous section, we will slightly abuse notation, denoting a differential form on M_C and its pullback to $M_a \times M_b$, via the canonical projection, by the same symbol. Similarly, given a vector field on M_C, we denote by the same symbol its horizontal lift to the product manifold $M_a \times M_b$.

Further, we denote by l^a and l^b the multi-brackets of $L_\infty(M_a, \omega_a)$ and $L_\infty(M_b, \omega_b)$ respectively, and by l the multi-brackets of $L_\infty(M, \omega)$.

6.6.1 The construction of H and its properties

The source of H is $L_\infty(M_a, \omega_a) \oplus L_\infty(M_b, \omega_b)$, which, being a direct sum of L_∞-algebras,is itself an L_∞-algebra. We spell this out, assuming $n_b \geq n_a$. The underlying complex is

$$C^\infty(M_b) \to \cdots \to C^\infty(M_a) \oplus \Omega^{n_b - n_a}(M_b) \to \cdots \to \Omega^{n_a - 1}(M_a) \oplus \Omega^{n_b - 1}(M_b).$$

Its multibrackets l_k^{ab} (for $k \geq 1$) are defined by

$$l_k^{ab}\left(\alpha_1 \oplus \beta_1, \ldots, \alpha_k \oplus \beta_k\right) = l_k^a\left(\alpha_1, \ldots, \alpha_k\right) \oplus l_k^b\left(\beta_1, \ldots, \beta_k\right)$$

where $\alpha_1 \oplus \beta_1, \ldots, \alpha_k \oplus \beta_k \in L_\infty(M_a, \omega_a) \oplus L_\infty(M_b, \omega_b)$. Notice that $L_\infty(M_a, \omega_a) \oplus L_\infty(M_b, \omega_b)$ is a Lie N-algebra, where $N := Max\{n_a, n_b\}$, while $L_\infty(M, \omega)$ - the target of H - is a Lie $(n_a{+}n_b{+}1)$-algebra.

We now argue that there is a natural candidate for the first component of an L_∞-morphism as in (6.100).

Given $\alpha \in \Omega^{n_a-1}_{\mathrm{Ham}}(M)\, M_a$ and $\beta \in \Omega^{n_b-1}_{\mathrm{Ham}}(M)\, M_b$, take Hamiltonian vector fields X_α and X_β for them, and consider $X_\alpha + X_\beta$ on $M_a \times M_b$. It is again a Hamiltonian vector field, since

$$\iota_{(X_\alpha + X_\beta)}\omega = -d\left[\alpha \wedge \omega_b + \omega_a \wedge \beta\right].$$

Hence there is a well-defined map

$$\begin{aligned} h : \Omega^{n_a-1}_{\mathrm{Ham}}(M_a) \oplus \Omega^{n_b-1}_{\mathrm{Ham}}(M_b) &\to \Omega^{n_a+n_b}_{\mathrm{Ham}}(M) \\ \alpha \oplus \beta &\mapsto \alpha \wedge \omega_b + \omega_a \wedge \beta \,. \end{aligned} \tag{6.101}$$

Endow $\Omega^{n_a-1}_{\mathrm{Ham}}(M_a) \oplus \Omega^{n_b-1}_{\mathrm{Ham}}(M_b)$ with the bracket l_2^{ab}, i.e., the sum of the binary brackets l_2^a and l_2^b on the two factors. Denoting all binary brackets by $\{\cdot,\cdot\}$ to ease the notation, we have

$$h\Big(\Big\{\alpha_1 \oplus \beta_1, \alpha_2 \oplus \beta_2\Big\}\Big) = \Big\{h(\alpha_1 \oplus \beta_1)\ ,\ h(\alpha_2 \oplus \beta_2)\Big\} + (-1)^{n_a} d\left[\alpha_1 \wedge d\beta_2 - \alpha_2 \wedge d\beta_1\right] .$$

That is, h does not preserve the binary brackets on the nose, but just up to an exact term. This a characteristic feature of the first component of an L_∞-morphism. Indeed, in Thm. 6.6.2 we extend h to an L_∞-morphism from $L_\infty(M_a,\omega_a) \oplus L_\infty(M_b,\omega_b)$ to $L_\infty(M,\omega)$. The concrete expression of the L_∞-morphism is motivated by the results of Section 6.4 and in particular by Theorem 6.4.3.

We will use the square brackets introduced in Def. 6.1.10, for $C = a, b$. Recall that $[\dots]^C_k$ is defined for all $k \geq 0$, and that it vanishes unless all entries have degree zero (i.e., are Hamiltonian forms). Recall also that $[\,1]^C_0 = -\omega_C$ and that for $k \geq 1$, by Remark 6.1.9,

$$[\alpha_1,\dots,\alpha_k]^C_k = \{\alpha_1,\dots,\alpha_k\}_C - \delta_{k,1} d_C \alpha_1\,, \quad C = a, b\,,$$

where $\{\alpha_1,\dots,\alpha_k\}_C$ is the k-bracket of $L_\infty(M_C,\omega_C)$ and d_C is the de Rahm differential on M_C.

Theorem 6.6.2. *Let (M_C,ω_C) be n_C-plectic manifolds. There is an L_∞-morphism:*

$$H\colon L_\infty(M_a,\omega_a) \oplus L_\infty(M_b,\omega_b) \rightsquigarrow L_\infty(M_a \times M_b, \omega_a \wedge \omega_b)$$

whose first component is injective. The components of H will be denoted by H_l ($l \geq 1$). They are determined by graded skew-symmetry and the requirement that:

$$\begin{aligned} H_{k+m}(\alpha_1,\dots,\alpha_k,\beta_1,\dots,\beta_m) = \;& t^a_{m,|\alpha_1|}\delta_{k,1}\alpha_1 \wedge [\beta_1,\dots,\beta_m]^b_m & (6.102)\\ +\;& t^b_{k,|\beta_1|}\delta_{m,1}[\alpha_1,\dots,\alpha_k]^a_k \wedge \beta_1\,, & (6.103) \end{aligned}$$

where $k+m \geq 1$, $\alpha_1,\dots,\alpha_k \in L_\infty(M_a,\omega_a)$, $\beta_1,\dots,\beta_m \in L_\infty(M_b,\omega_b)$, $[\,1\,]^C_0 = -\omega_C$ and the coefficients are, for all $i \leq 0$:

$$\begin{aligned} t^a_{m,i} &= -\frac{1}{2}(-1)^{m(n_a+1+i)}\,, \qquad m \geq 1 \\ t^b_{k,i} &= -\frac{1}{2}(-1)^{i(n_a+1)+k}\,, \qquad k \geq 1 \end{aligned} \tag{6.104}$$

and:

$$t^a_{0,i} = -1, \qquad t^b_{0,i} = -(-1)^{i(n_a+1)}\,.$$

Above, δ denotes the Kronecker delta, and $|\alpha_1|$ refers to the degree[15] *of α_1 as an element of $L_\infty(M_a,\omega_a)$.*

[15]This differs by $n_a - 1$ from the degree of α_1 as a differential form.

Remark 6.6.3. Notice that H, applied to a family of elements lying in $(L_\infty(M_a,\omega_a)\oplus\{0\})\cup(\{0\}\oplus L_\infty(M_b,\omega_b))$, vanishes unless: either exactly one element is of the form $\alpha\oplus 0$ and the remaining elements have degree zero, or exactly one element is of the form $0\oplus\beta$ and the remaining elements have degree zero.

Remark 6.6.4. The first component H_1 is clearly injective for it is given by

$$H_1(\alpha)=\alpha\wedge\omega_b \qquad\text{and}\qquad H_1(\beta)=(-1)^{|\beta|(n_a+1)}\,\omega_a\wedge\beta,$$

where $\alpha\in L_\infty(M_a,\omega_a)$ and $\beta\in L_\infty(M_b,\omega_b)$.

The restriction of H_1 to $L_\infty(M_a,\omega_a)\oplus\{0\}$ is a strict morphism. This can be seen using Remark 6.6.3, since the higher components of H vanish if all entries lie in $L_\infty(M_a,\omega_a)\oplus\{0\}$, or alternatively it can be seen directly using Lemma 6.6.7 below. The same holds for the restriction of H to $\{0\}\oplus L_\infty(M_b,\omega_b)$.

Remark 6.6.5. Recall that the composition $\psi\circ\phi$ of two L_∞-morphisms is given by $(\psi\circ\phi)_k=\sum_{l=1}^k\sum_{k_1+\cdots+k_l=k}\pm\psi_l\circ(\phi_{k_1}\otimes\cdots\otimes\phi_{k_l})$. Possibly up to signs, the L_∞-morphism H given Thm. 6.6.2 has the following property: for any action of a Lie group G_C on (M_C,ω_C) with homotopy moment map f^C $(C=a,b)$, one has

$$F=H\circ(f^a\oplus f^b),$$

where F is the homotopy moment map constructed in Thm. 6.4.3 out of f^a and f^b. In other words, the diagram (6.3) commutes.

Example 6.6.6. Let $n_a=n_b=1$. That is, (M_a,ω_a) and (M_b,ω_b) are symplectic manifolds, and so (M,ω) is a 3-plectic manifold. Consequently, the cochain complex L underlying the Lie 3-algebra $L_\infty(M,\omega)$ is

$$C^\infty(M)\to\Omega^1(M)\to\Omega^2(M)\to\Omega^3_{\mathrm{Ham}}(M)\ .$$

On the other hand, $L_\infty(M_a,\omega_a)\oplus L_\infty(M_b,\omega_b)=C^\infty(M_a)\oplus C^\infty(M_b)$ is just a Lie-algebra. The higher components of the L_∞-embedding of theorem 6.6.2 read

$$\mathrm{H}_2(f_a\oplus f_b,g_a\oplus g_b)=\frac{1}{2}(f_a\wedge dg_b-df_a\wedge g_b-g_a\wedge df_b+dg_a\wedge f_b)\ ,$$

$$\begin{aligned}\mathrm{H}_3(f_a\oplus f_b,g_a\oplus g_b,h_a\oplus h_b)=\frac{1}{2}&(f_a\{g_b,h_b\}_2+f_b\{g_a,h_a\}_2-g_a\{f_b,h_b\}_2 &(6.105)\\ &-g_b\{f_a,h_a\}_2+h_a\{f_b,g_b\}_2+h_b\{f_a,g_a\}_2)\ , &(6.106)\end{aligned}$$

for all $f_C,g_C,h_C\in C^\infty(M_C)$, $C=a,b$. Notice that since $L_\infty(M_a,\omega_a)\oplus L_\infty(M_b,\omega_b)$ is a Lie algebra, we can use formulae (6.25) and (6.27) to double-check that H is indeed an L_∞-morphism.

6.6.2 The proof

We now turn to the proof of Thm. 6.6.2. We will use repeatedly the following Lemma.

Lemma 6.6.7. *For all $\alpha_1,\dots,\alpha_k\in L_\infty(M_a,\omega_a)$ and $\beta_1,\dots,\beta_m\in L_\infty(M_b,\omega_b)$, where $k,m\geq 0$ and $k+m\geq 1$, we have*

$$[\alpha_1\omega_b,\dots,\alpha_k\omega_b,\omega_a\beta_1,\dots,\omega_a\beta_m]_{k+m}=-(-1)^{m(n_a+1)}[\alpha_1,\dots,\alpha_k]_k\wedge[\beta_1,\dots,\beta_m]_m\,.$$

Proof. We may assume that all the α and β have degree zero, for otherwise the equation is trivially satisfied. It is straightforward to verify that the Hamiltonian vector field of $\alpha\omega_b$ (w.r.t $\omega_a\wedge\omega_b$) equals the Hamiltonian vector field X_α of α (w.r.t ω_a), and the exactly analogous statement holds for $\omega_a\beta$. The statement of the lemma follows from

$$\iota(X_{\alpha_1}\wedge X_{\alpha_2}\wedge\cdots\wedge X_{\beta_m})(\omega_a\wedge\omega_b)=(-1)^{m(n_a+1-k)}\iota(X_{\alpha_1}\wedge\cdots\wedge X_{\alpha_k})\omega_a\wedge\iota(X_{\beta_1}\wedge\cdots\wedge X_{\beta_m})\omega_b$$

together with the identity $\varsigma(k)\varsigma(m)\varsigma(k+m)=-(-1)^{km}$. □

According to the conditions that an L_∞-morphism has to obey (see for instance [20, Def. 2.4]), we have to check that the following relation holds for all $N \in \mathbb{N}_{>0}$ and for all $\vec{x} = (x_1, \ldots, x_N) \in (L_\infty(M_a, \omega_a) \oplus L_\infty(M_b, \omega_b))^{\otimes N}$:

$$\sum_{i+j=N+1} (-1)^{i(j-1)} \sum_{\sigma \in \mathrm{Sh}_{i,j-1}} (-1)^\sigma \epsilon(\sigma, \vec{x})\, H_j\left(l_i^{ab}\left(x_{\sigma(1)}, \ldots, x_{\sigma(i)}\right), x_{\sigma(i+1)}, \ldots, x_{\sigma(N)}\right) \tag{6.107}$$
$$= \sum_{\ell=1}^{N} \sum_{\substack{N_1+\cdots+N_\ell=N \\ N_1 \le \cdots \le N_\ell}} (-1)^{\gamma(\ell,\vec{N})} \sum_{\sigma \in Sh^{<}_{N_1,\ldots,N_\ell}} (-1)^\sigma \epsilon(\sigma, \vec{x}) \epsilon\left(\rho, \vec{H}\right)$$
$$l_\ell\left(H_{N_1}(x_{\sigma(1)}, \ldots, x_{\sigma(N_1)}), \ldots, H_{N_\ell}(x_{\sigma(N-N_\ell+1)}, \ldots, x_{\sigma(N)})\right).$$

Here

- $\gamma(\ell, \vec{N}) \equiv \frac{\ell(\ell-1)}{2} + N_1(\ell-1) + N_2(\ell-2) + \cdots + N_{\ell-1}$.
- $Sh^{<}_{N_1,\ldots,N_\ell} \subset Sh_{N_1,\ldots,N_\ell}$ is the set of $(N_1, \ldots, N_\ell)$-unshuffles such that

$$\sigma(N_1 + \cdots + N_{i-1} + 1) < \sigma(N_1 + \cdots + N_{i-1} + N_i + 1) \quad \text{whenever } N_i = N_{i+1}.$$

- $\vec{H} = \left(H_{N_1}, \ldots, H_{N_\ell}, x_{\sigma(1)}, \ldots, x_{\sigma(N)}\right)$ and ρ is the permutation of $\{1, \ldots, \ell+N\}$ sending $\vec{H}$ to $\left(H_{N_1}, x_{\sigma(1)}, \ldots, x_{\sigma(N_1)}, \ldots, H_{N_\ell}, x_{\sigma(N-N_\ell+1)}, \ldots, x_{\sigma(N)}\right)$.

As usual, $(-1)^\sigma$ denotes the sign of the permutation σ and $\epsilon(\sigma, \vec{x})$ denotes the Koszul sign.

Remark 6.6.8. Notice that on the l.h.s. of eq. (6.107), the sign of the summand corresponding to $i = N, j = 1$ is $+1$ (since the only permutation appearing is the identity).

On the r.h.s., the sign of the summand corresponding to $l = N$ is $+1$. Indeed $N_1 = \cdots = N_l = 1$, so that $\gamma(\ell, \vec{N}) = +1$, $\sigma = id$, and all H_{N_i} have degree zero. Further, the sign of the summand corresponding to $\ell = 1$ is also $+1$, since $\gamma(1, \vec{N}) = +1$, $\sigma = id$ and $\rho = id$.

Proof of Thm. 6.6.2. Let $C = a, b$. We first check that H_j has degree $1-j$. For $j = 1$ this is clear. For $j = k+m \ge 2$, we use that $[\ldots]^C_m$, as an operation on $L_\infty(M_C, \omega_C)$, has degree $2-m$. Hence, for instance, if the elements $\alpha_1, \beta_1, \ldots, \beta_m$ all have degree zero, then $H_{1+m}(\alpha_1, \beta_1, \ldots, \beta_m) = \pm\frac{1}{2}\alpha_1[\beta_1, \ldots, \beta_m]^b_m$ is the product of a $n_a - 1$ and $(n_b - 1) + (2 - m)$ form, that is, a $n_a + n_b - m$ form, which therefore is an element of $L_\infty(M_a \times M_b, \omega_a \wedge \omega_b)$ of degree $-m = 1 - (1+m) = 1 - j$.

The rest of the proof is devoted to checking that H is an L_∞-morphism. Our strategy is as follows. We propose an educated ansatz for H depending on some arbitrary parameters and then we will impose on it the L_∞-morphism conditions (6.107). Equations (6.107) will turn out to be an over-determined system of equations for the parameters of the ansatz, and we will show that a solution is given by eq. (6.104).

The ansatz is the following: for the first component of H,

$$H_1(\alpha) = s^a_{0,|\alpha|}\alpha \wedge (-\omega_b), \qquad H_1(\beta) = s^b_{0,|\beta|}(-\omega_a) \wedge \beta\,.$$

For the higher components of H, i.e. for $k + m \ge 2$, $H_{k+m}(\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_m)$ equals:

$$\frac{s^a_{m,|\alpha_1|}}{2}\delta_{k,1}\alpha_1 \wedge [\beta_1, \ldots, \beta_m]^b_m + \frac{s^b_{k,|\beta_1|}}{2}\delta_{m,1}[\alpha_1, \ldots, \alpha_k]^a_k \wedge \beta_1\,, \tag{6.108}$$

where $\alpha_1, \ldots, \alpha_k \in L_\infty(M_a, \omega_a)$ and $\beta_1, \ldots, \beta_m \in L_\infty(M_b, \omega_b)$ are homogeneous elements of their respective graded spaces. Here $s^a_{m,|\alpha_1|}$ depends on the number of β's and the degree of α_1. It cannot depend on the number of α's since if there is more than one the corresponding term in (6.108) is zero, and it cannot depend on the degree of the β's since if $|\beta_1 \otimes \cdots \otimes \beta_m| < 0$ then the corresponding term in (6.108) is again zero. A similar discussion applies to $s^b_{k,|\beta_1|}$.

We now apply condition (6.107) to our ansatz for H and elements $\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_m$. We are going to consider six different cases depending on k and m, namely $\{k \geq 1, m = 0\}$, $\{k = 0, m \geq 1\}$, $\{k = 1, m = 1\}$, $\{k > 1, m > 1\}$, $\{k = 1, m > 1\}$ and $\{k > 1, m = 1\}$. We will use repeatedly Remark 6.6.3 and the fact that for $i \geq 2$ the multibrackets l_i vanish unless all entries have degree zero.

Case $\{\mathbf{k} \geq \mathbf{1}, \mathbf{m} = \mathbf{0}\}$.

This case will allow us to calculate $s^a_{0,i}$, $i \leq 0$. The condition (6.107) evaluated on $\alpha_1, \dots, \alpha_k \in L_\infty(M_a, \omega_a)$ reads

$$H_1\left(l^a_k\left(\alpha_1, \dots, \alpha_k\right)\right) = l_k\left(H_1\left(\alpha_1\right), \dots, H_1\left(\alpha_k\right)\right) , \tag{6.109}$$

as one sees using Rem. 6.6.3, together with Remark 6.6.8 to determine the signs.
Using now that:

$$\begin{aligned} H_1\left(l^a_k\left(\alpha_1, \dots, \alpha_k\right)\right) &= -s^a_{0,2-k+|\alpha|}\, l^a_k\left(\alpha_1, \dots, \alpha_k\right) \wedge \omega_b , & (6.110)\\ l_k\left(H_1\left(\alpha_1\right), \dots, H_1\left(\alpha_k\right)\right) &= \left(-s^a_{0,|\alpha_1|}\right) \dots \left(-s^a_{0,|\alpha_k|}\right) l^a_k\left(\alpha_1, \dots, \alpha_k\right) \wedge \omega_b , & (6.111)\end{aligned}$$

where $|\alpha| = |\alpha_1 \otimes \cdots \otimes \alpha_k|$ and using Lemma 6.6.7 in the second equation when $k \geq 2$, we conclude that we can choose $s^a_{0,i} = -1$ for all $i \leq 0$.

Case $\{\mathbf{k} = \mathbf{0}, \mathbf{m} \geq \mathbf{1}\}$.

This case will allow as to calculate $s^b_{0,i}$, $i \leq 0$. The condition (6.107) evaluated on $\beta_1, \dots, \beta_m \in L_\infty(M_b, \omega_b)$, similarly to the case above, reads:

$$H_1\left(l^b_m\left(\beta_1, \dots, \beta_m\right)\right) = l_m\left(H_1\left(\beta_1\right), \dots, H_1\left(\beta_m\right)\right) . \tag{6.112}$$

Using now that:

$$\begin{aligned} H_1\left(l^b_m\left(\beta_1, \dots, \beta_m\right)\right) &= -s^b_{0,2-m+|\beta|}\, \omega_a \wedge l^b_m\left(\beta_1, \dots, \beta_m\right) , & (6.113)\\ l_m\left(H_1\left(\beta_1\right), \dots, H_1\left(\beta_m\right)\right) &= \left(-s^b_{0,|\beta_1|}\right) \dots \left(-s^b_{0,|\beta_m|}\right)(-1)^{m(n_a+1)}\, \omega_a \wedge l^b_m\left(\beta_1, \dots, \beta_m\right) , & (6.114)\end{aligned}$$

where $|\beta| = |\beta_1 \otimes \cdots \otimes \beta_m|$ and using Lemma 6.6.7 in the second equation when $m \geq 2$, we conclude (taking $m = 1$) that equation (6.109) implies:

$s^b_{0,1+|\beta|} = (-1)^{(n_a+1)} s^b_{0,|\beta|}$

and therefore $s^b_{0,i} = (-1)^{i(n_a+1)}\, s^b_{0,0}$, $\quad i \leq 0$. Plugging this into into eq. (6.112) it can be easily verified that eq. (6.112) is solved by:

$$s^b_{0,i} = -\left(-1\right)^{i(n_a+1)} , \quad i \leq 0 .$$

Case $\{\mathbf{k} = \mathbf{1}, \mathbf{m} = \mathbf{1}\}$.

This case will allow as to find $s^a_{1,i}$ and $s^b_{1,i}$ for $i \leq 0$. The condition (6.107) evaluated on α, β, where $\alpha \in L_\infty(M_a, \omega_a)$ and $\beta \in L_\infty(M_b, \omega_b)$, reads:

$$-H_2\left(l^a_1(\alpha), \beta\right) - (-1)^{|\alpha|} H_2\left(\alpha, l^b_1(\beta)\right) = l_1\left(H_2(\alpha, \beta)\right) + l_2\left(H_1(\alpha), H_1(\beta)\right) . \tag{6.115}$$

(The l.h.s. corresponds to the summand $i = 1, j = 2$ in (6.107), and the signs for the r.h.s. follow from Remark 6.6.8.) Recall that by ansatz (6.108), for all $A \in L_\infty(M_a, \omega_a)$ and $B \in L_\infty(M_b, \omega_b)$ we have

$$H_2(A,B) = \frac{s^a_{1,|A|}}{2} A \wedge [B]^b_1 + \frac{s^b_{1,|B|}}{2}\, [A]^a_1 \wedge B.$$

In order to solve equation (6.115) we have to analyze the different cases in terms of the degree of α and β. If $|\alpha| = |\beta| = 0$ the l.h.s. of (6.115) is zero while the r.h.s. is

$$-\frac{s^a_{1,0}}{2} d\alpha \wedge d\beta - (-1)^{n_a} \frac{s^b_{1,0}}{2} d\alpha \wedge d\beta + (-1)^{n_a} d\alpha \wedge d\beta\,,$$

as one sees using Lemma 6.6.7. Hence we can take:

$$s^a_{1,0} = (-1)^{n_a}\,, \quad s^b_{1,0} = 1. \tag{6.116}$$

Now, if $|\alpha| = 0$ and $|\beta| < 0$ the first and fourth term in equation (6.115) vanish, and that equation translates into

$$-\frac{s^b_{1,|\beta|+1}}{2}\, [\alpha]^a_1 \wedge l^b_1(\beta) = (-1)^{n_a} \frac{s^b_{1,|\beta|}}{2}\, [\alpha]^a_1 \wedge l^b_1(\beta)\,,$$

implying that $s^b_{1,|\beta|+1} = (-1)^{n_a+1} s^b_{1,|\beta|}$. Together with equation (6.116) this implies finally that

$$s^b_{1,i} = (-1)^{i(n_a+1)}\,, \qquad i \leq 0\,.$$

By means of a completely analogous calculation for the case $|\alpha| < 0$ and $|\beta| = 0$ we obtain $s^a_{1,|\alpha|+1} = -s^a_{1,|\alpha|}$, so we can choose

$$s^a_{1,i} = (-1)^{n_a+i}\,, \qquad i \leq 0\,.$$

Lastly, the case $|\alpha| < 0$ and $|\beta| < 0$ is trivial since both sides of equation (6.115) vanish.

Case $\{\mathbf{k} > \mathbf{1}, \mathbf{m} > \mathbf{1}\}$.

This case will allow as to find $s^a_{k,i}$ and $s^b_{m,i}$ for $i \leq 0$ and $k, m > 1$. The condition (6.107) evaluated on $(\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_m)$, where $\alpha_1, \ldots, \alpha_k \in L_\infty(M_a, \omega_a)$ and $\beta_1, \ldots, \beta_m \in L_\infty(M_b, \omega_b)$, reduces to

$$\begin{aligned}&(-1)^{km} H_{m+1}\left(l^a_k(\alpha_1, \ldots, \alpha_k), \beta_1, \ldots, \beta_m\right) + (-1)^k H_{k+1}\left(\alpha_1, \ldots, \alpha_k, l^b_m(\beta_1, \ldots, \beta_m)\right) \\ =& l_{k+m}\left(H_1(\alpha_1), \ldots, H_1(\alpha_k), H_1(\beta_1), \ldots, H_1(\beta_m)\right)\,,\end{aligned} \tag{6.117}$$

where in the l.h.s. only the summands corresponding to $i = k$ and $i = m$ appear by Rem. 6.6.3, and for the r.h.s. we use Remark 6.6.8 to determine the signs (a term involving l_1 does not appear, again due to Rem. 6.6.3).

From definition 6.1.12 it can be seen that equation (6.117) is only non-trivial if[16] $|\alpha| = |\beta| = 0$. Therefore, we will assume henceforth that this is the case. The two terms on the l.h.s. of equation (6.117) can be written as follows:

$$H_{m+1}\left(l^a_k(\alpha_1, \ldots, \alpha_k), \beta_1, \ldots, \beta_m\right) = \frac{s^a_{m,2-k}}{2}\, [\alpha_1, \ldots, \alpha_k]^a_k \wedge [\beta_1, \ldots, \beta_m]^b_m\,, \tag{6.118}$$

$$H_{k+1}\left(\alpha_1, \ldots, \alpha_k, l^b_m(\beta_1, \ldots, \beta_m)\right) = \frac{s^b_{k,2-m}}{2}\, [\alpha_1, \ldots, \alpha_k]^a_k \wedge [\beta_1, \ldots, \beta_m]^b_m\,. \tag{6.119}$$

[16] The fact that necessarily $|\alpha| = 0$ was already used to determine the sign of the second term on the l.h.s. above.

By Lemma 6.6.7, the r.h.s. of equation (6.117) can be written as:

$$l_{k+m}\left(H_1(\alpha_1),\dots,H_1(\alpha_k),H_1(\beta_1),\dots,H_1(\beta_m)\right)=-(-1)^{m(n_a+1)}\left[\alpha_1,\dots,\alpha_k\right]^a_k\wedge\left[\beta_1,\dots,\beta_m\right]^b_m\,. \tag{6.120}$$

From the last three equations we obtain:

$$(-1)^{km}\frac{s^a_{m,2-k}}{2}+(-1)^k\frac{s^b_{k,2-m}}{2}=-(-1)^{m(n_a+1)}\,,$$

which is solved by

$$s^a_{m,i}=-(-1)^{m(n_a+i+1)}\,,\qquad s^b_{k,i}=-(-1)^{i(n_a+1)+k}\,,\qquad m,k>1\,,\ i\leq 0\,.$$

Cases $\{\mathbf{k=1,m>1}\}$ and $\{\mathbf{k>1,m=1}\}$.

Notice that this point we have already explicitly solved all the parameters $s^a_{m,i}$ and $s^b_{k,i}$ for all $k,m\geq 0$ and $i\leq 0$. Although this was obtained by separately analyzing different cases given by different values of k and m, the result be summarized in a single formula, namely

$$s^a_{m,i}=-(-1)^{m(n_a+i+1)}\,,\qquad s^b_{k,i}=-(-1)^{i(n_a+1)+k}\,,\qquad m,k\geq 0\,,\ i\leq 0\,. \tag{6.121}$$

However, there remain two cases to be solved, namely $\{k=1,m>1\}$ and $\{k>1,m=1\}$. Notice that we do not have any parameter left to be fixed, so checking those cases is really a constraint.

We consider first the case $\{k=1,m>1\}$. At first, we also assume $m>2$. The condition (6.107) evaluated on $(\alpha,\beta_1,\dots,\beta_m)$ reads

$$\begin{aligned}(-1)^m H_{m+1}\left(l^a_1(\alpha),\beta_1,\dots,\beta_m\right)+\sum_{1\leq p<q\leq m}(-1)^{p+q}H_m\left(\alpha,l^b_2(\beta_p,\beta_q),\beta_1,\dots,\widehat{\beta_p},\dots,\widehat{\beta_q},\dots\beta_m\right)\\ +H_2((l^b_m(\beta_1,\dots,\beta_m),\alpha)\\ = l_{m+1}\left(H_1(\alpha),H_1(\beta_1),\dots,H_1(\beta_m)\right)+l_1(H_{m+1}(\alpha,\beta_1,\dots,\beta_m))\,.\end{aligned} \tag{6.122}$$

(On the l.h.s. the first term corresponds to $i=1$ in eq. (6.107), the second to $i=2$, and the third to $i=m$; not other values of i contribute by Remark 6.6.3. On the r.h.s. only the terms corresponding to l_{m+1} and l_1 appear since the multibrackets of $L_\infty(M_a\times M_b)$ with two or more entries vanish unless all the entries have degree zero, and the signs are given by Remark 6.6.8.) We may assume[17] $|\beta_1|=\dots=|\beta_m|=0$, for otherwise both sides of the above equation vanish by Remark 6.6.3.

The first term on the l.h.s. of eq. (6.122) reads

$$(-1)^m\frac{s^a_{m,|\alpha|+1}}{2}l^a_1(\alpha)\wedge[\beta_1,\dots,\beta_m]\,. \tag{6.123}$$

The second term on the l.h.s. equals

$$\frac{s^a_{m-1,|\alpha|}}{2}\alpha\wedge d[\beta_1,\dots,\beta_m]\,. \tag{6.124}$$

[17]This assumption was already used to determine the sign of the second term on the l.h.s. above.

To see this, we use the computation

$$\begin{aligned}
&\sum_{1\leq p<q\leq m} (-1)^{p+q}[l_2^b(\beta_p,\beta_q),\beta_1,\dots,\widehat{\beta_p},\dots,\widehat{\beta_q},\dots,\beta_m] \\
=&\varsigma(m-1)\sum_{1\leq p<q\leq m} (-1)^{p+q}\iota(X_{l_2^b(\beta_p,\beta_q)}\wedge X_{\beta_1}\wedge\cdots\wedge\widehat{X_{\beta_p}}\wedge\cdots\wedge\widehat{X_{\beta_q}}\wedge\cdots\wedge X_{\beta_m})\omega_b \\
=&\varsigma(m-1)(-1)^m d\iota(X_{\beta_1}\wedge\cdots\wedge X_{\beta_m})\omega_b \\
=&d[\beta_1,\dots,\beta_m].
\end{aligned} \tag{6.125}$$

where we used [18, Lemma 9.2] in the second equality and $\varsigma(m-1)\varsigma(m)=(-1)^m$.

The third term on the l.h.s. reads

$$-\frac{s^b_{1,2-m}}{2}[\alpha]\wedge[\beta_1,\dots,\beta_m]-\frac{s^a_{1,|\alpha|}}{2}\alpha\wedge[l^b_m(\beta_1,\dots,\beta_m)], \tag{6.126}$$

where the second summand vanishes because of the assumption $m>2$.

The first term on the r.h.s. of eq. (6.122), using Lemma 6.6.7 and $-s^A_{0,0}=-s^b_{0,0}=1$, equals

$$-(-1)^{m(n_a+1)}[\alpha]\wedge[\beta_1,\dots,\beta_m]\,. \tag{6.127}$$

The last term on the r.h.s. is

$$\frac{s^a_{m,|\alpha|}}{2}l_1(\alpha\wedge[\beta_1,\dots,\beta_m])=\frac{s^a_{m,|\alpha|}}{2}\Big(d\alpha\wedge[\beta_1,\dots,\beta_m]+(-1)^{n_a-1-|\alpha|}\alpha\wedge d[\beta_1,\dots,\beta_m]\Big)\,. \tag{6.128}$$

The term in (6.124) cancels out with the second summand in (6.128). Further, using that $l_1\alpha-[\alpha]=d\alpha$ by Remark 6.1.9 and the fact that $[\alpha]$ vanishes if $|\alpha|\neq 0$, one check that the term (6.123) minus one half the term (6.127) equals $\frac{s^a_{m,|\alpha|}}{2}d\alpha\wedge[\beta_1,\dots,\beta_m]$, which is exactly the first summand in eq. (6.128). Finally, the term (6.126) cancels out with one half the term (6.127).

Now, if $k=1,m=2$, then the term (6.124) is omitted (because the summand $i=2$ on the l.h.s. of condition (6.107) is already given by the term (6.126)), and in (6.126) the second summand no longer vanishes. We conclude that the case $\{k=1,m>1\}$ indeed works out with the choice of parameters given in (6.121).

One check in a similar way that the same holds for the case $\{k>1,m=1\}$. This concludes the proof that H, as defined in the statement of the theorem, is an honest L_∞-morphism.

□

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