An introduction to the limit set of Kleinian groups

escrito por

JORGE YÁÑEZ ESCANCIANO

Tutor: FRANCISCO JAVIER CIRRE TORRES

Facultad de Ciencias
UNIVERSIDAD DE EDUCACIÓN A DISTANCIA

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Abstract

Resumen:

En este trabajo se analiza el conjunto límite de los grupos Kleinianos. Asumiendo conocimientos genericos, se describen las transformaciones de Möbius y los grupos Kleinianos y se enuncian algunas de sus propiedades. Ulteriormente, se caracteriza el conjunto límite de los grupos Kleinianos examinando sus propiedades básicas y topológicas asi como su convergencia e invariancia. Frecuentemente, el conjunto limite resulta ser un fractal cuyas principales propiedades son analizadas haciendo enfasis en su estructura local.

Abstract:

In this work the limit set of Kleinian groups is analyzed. Assuming basic knowledge, the Möbius transformations and the Kleinian groups are described outlining some of their properties. Subsequently, the limit set of Kleinian groups is characterized considering its basic and topological properties as well as its convergence and invariance. Frequently, the limit set results to be a fractal whose main properties are analyzed stressing the investigation of its local structure.

Keywords: Grupos Kleinianos, Conjunto Límite, Fractales

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CHAPTER 1

PREFACE

The Beauty of the Limit Set

In their excellent work *Indra's Pears*(2002), the Field's medalist D. Mumford together with his co-authors, Series and Wright, have shown some fascinating images. Those correspond to limit sets created by the recursive application of Möbius transformations to a set of circles located in some particular positions.

The transformation that those authors utilized two disks, D_A and D_a, see Figure 1.1, in the way indicated by the arrows of the figure. The transformation operates in the following way: i) the disks D_a and D_A do not overlap; ii) the outside of the disk D_A corresponds to the inside of D_a; iii) the inside of D_A is mapped in the outside of D_a; iv) the circles are mapped one in the other.

In the Figure, the successive application of the transformation, provides the images, circles, marked in yellow and red.

We may slightly complicate the pattern above. The combination of two transformations can be applied to a set of two pairs of circles as depicted in Figure 1.2. The combination of the transformations occurs as shown by the arrows. The outside of disk D_A is mapped inside circle D_a. The interior of D_A is mapped outside of D_A. The pairing procedure is analogous for both circles D_B and D_b.

The combination of the two transformations results in an interesting pattern. In Figure 1.3, four levels of circles and their transforms are represented. Each level is depicted in a different color. Notably, the circles start to accumulate in some locations, whilst other regions of the figure contain no images at all.

We may also consider the recursive and successive application of the transformations to the set of four tangent circles. We assume now that a sufficiently large number of stages, N, were calculated. The successive sets of circles obtained as images shrink to very small sizes. They

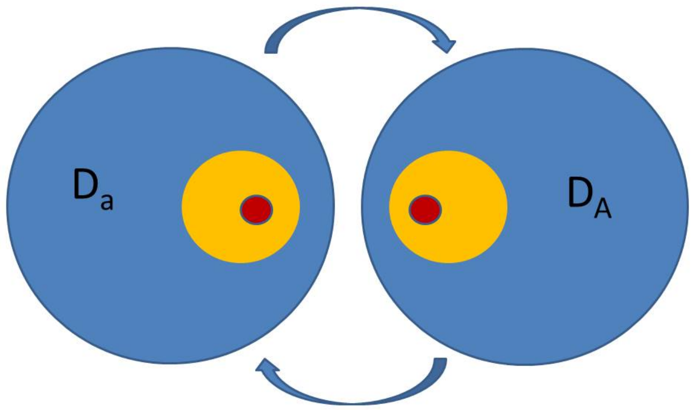


Figure 1.1: Circle pairing

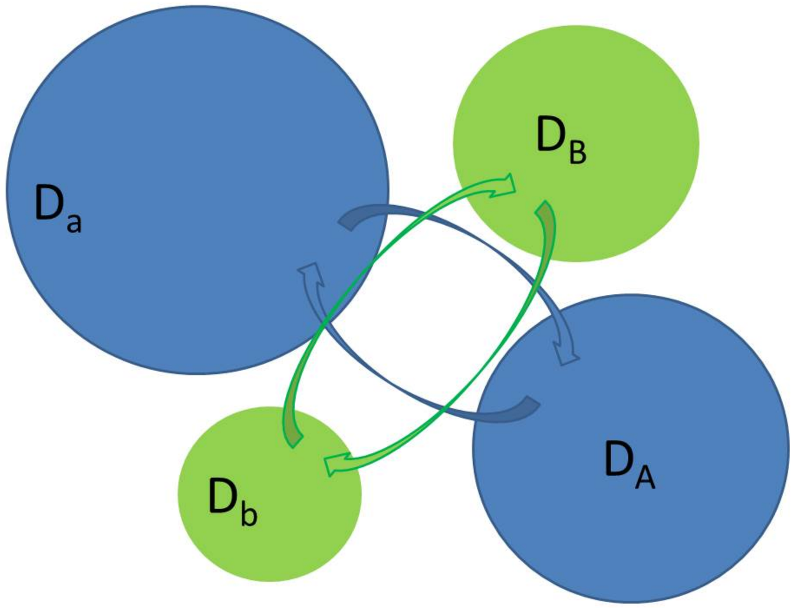


Figure 1.2: Circle pairing pairing when two pairs of disks are selected.

form a quasi one-dimensional line that can be interpreted as a limit set. The limit set is in this case continuous. Its topology appears to be of a fractal nature, as can be seen in the Figure 1.4. Notably, it contains some points in which the line suffers an abrupt change of direction in which tangents cannot be defined. It also tends to repeat its structure in different scales.

The enormous aesthetic beauty and the irresistible fascinating power of these images amply justify the fact D. Mumford has devoted them a monographic book, even if recreational.

The attractive power of the images obtained is not new. Surprisingly, the picture above was

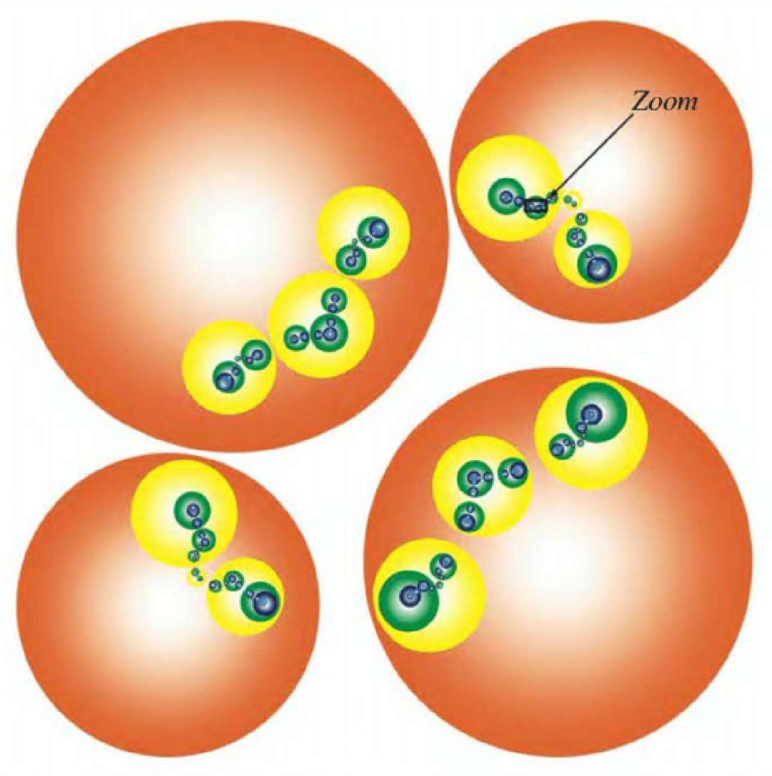


Figure 1.3: Pattern obtained after applying the transformation four times. Image taken from Mumford et al. (2002)

not originally obtained in the computer's era, when images of fractals have become popular between enthusiasts of mathematics.

In spite of its complexity, its original creation happened in the XIX century. Figure 1.5 is in fact due to Fricke and Klein (1897), and can be found in one of the numerous reprints of their seminal work, Fricke and Klein (1965).

It is the hope of the author, that previous paragraphs have attracted the curiosity of wise readers raising some questions. For example: What is the nature of the limit set that appears in such a simple but fascinating way in Figure 1.4? In addition to its absorbing beauty, what are its characteristics and its dimensions?

The pages of this work are humbly devoted to a very limited and partial answer of these questions.

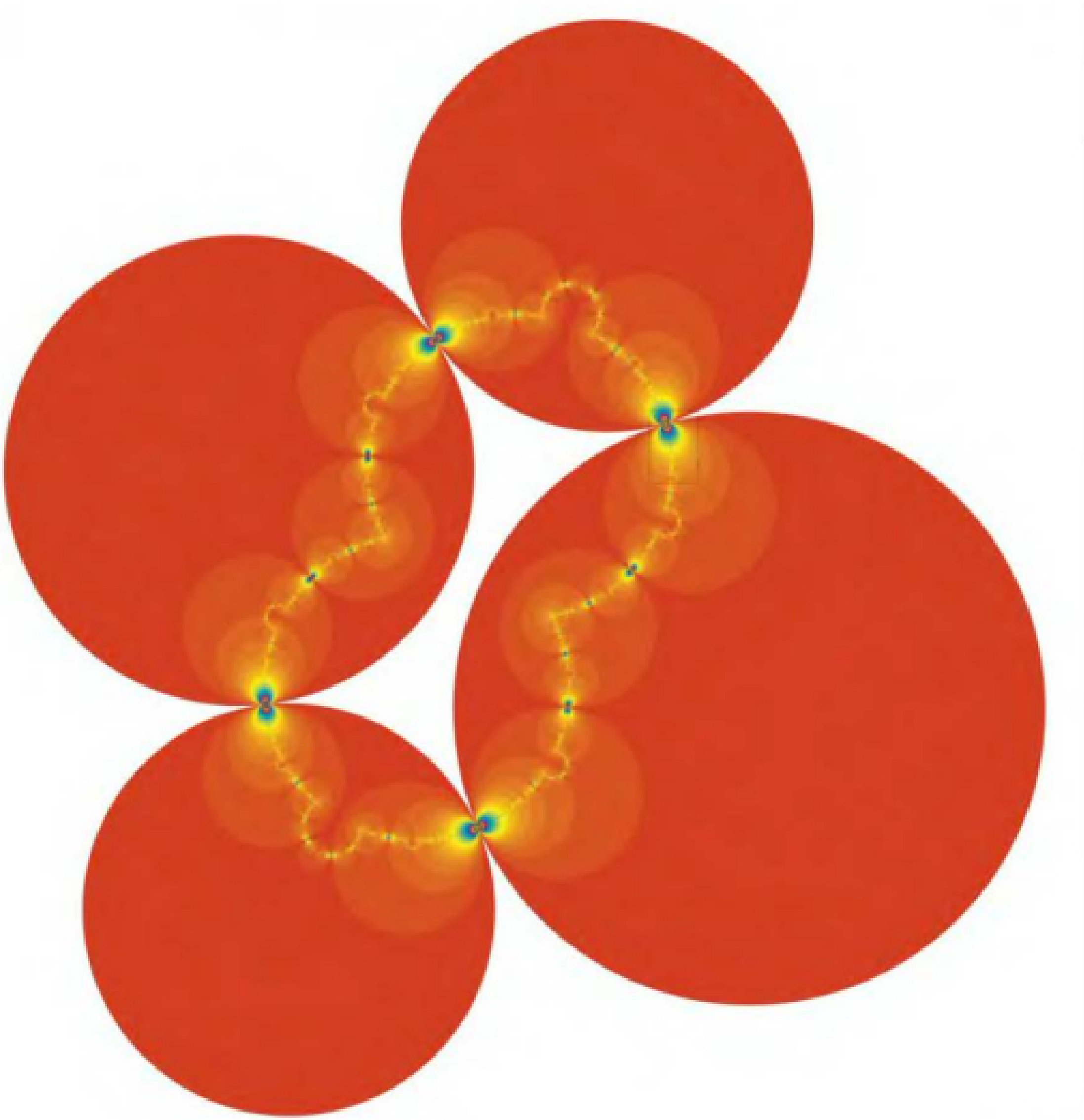


Figure 1.4: Limit set of the transformation. Image taken from Mumford et al. (2002)

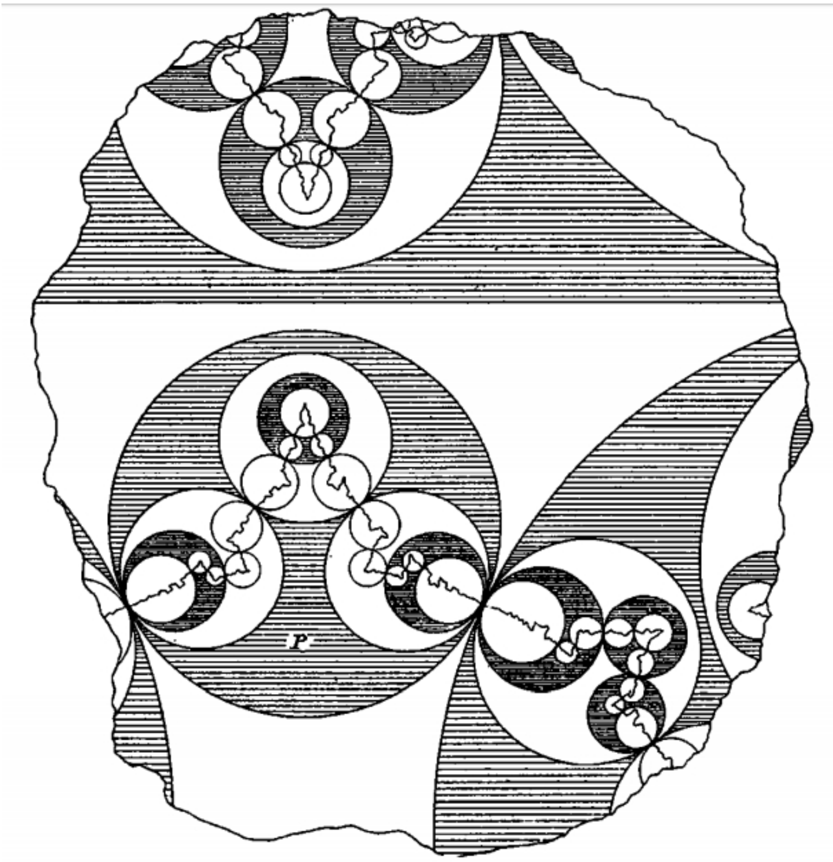


Figure 1.5: Limit set of the transformation as depicted by Fricke and Klein (1897)

CHAPTER 2

INTRODUCTION

In this chapter, we address some concepts of basic character. We will firstly introduce the Möbius Transformations describing also their formulation as matrices. Afterwards we will consider the classification of the Möbius Transformations. Also, we will briefly discuss some interesting topics such as the fixed points and the system of generators of the Möbius Transformations. Those will be necessary for the further comprehension of the topics analyzed in this work.

2.1 Möbius Transformations

2.1.1 General

We denote the extended complex plane $\mathbb{C} \cup \infty$ by $\hat{\mathbb{C}}$. Naturally, $\hat{\mathbb{C}}$ can also be regarded as a sphere which we call the Riemann Sphere and denote by $\mathbb{S}^2$.

This equivalence can be reached very simply. A plane passing through the equator of the sphere is stereographicaly projected into the sphere itself using a line traversing the north pole.

We will not further expound on this classic topic, referring the readers to the classic work of Ahlfors (1953).

Definition 1. *The transformations of* $\hat{\mathbb{C}}$ *of the form*

$$T(z) = \frac{az+b}{cz+d}, \tag{2.1}$$

and

$$T(\infty) = \frac{a}{c} \tag{2.2}$$

where $a, b, c, d \in \mathbb{C}$ *and* $ad - bc \neq 0$ *are called* Möbius Transformations

It is a very well-known result, that

Theorem 1. *$Aut(\hat{\mathbb{C}})$ are the Möbius Transformations, where $Aut(\hat{\mathbb{C}})$ stands for the groups of conformal automorphism of $\hat{\mathbb{C}}$*

The proof of this basic result is not included here. It can be found in classic texts on complex functions, like Jones and Singerman (1987, page 17).

Analogously to the Möbius transformations, we define a very similar mapping.

Definition 2. *The orientation reversing conformal homeomorphisms of $\hat{\mathbb{C}}$ are the transformations of the form,*

$$g(z) = \frac{a\bar{z}+b}{c\bar{z}+d}, \tag{2.3}$$

*where $ad - bc \neq 0$. Such transformations are called **Fractional reflections**.*

2.1.2 The Matrix Formulation of Möbius Transformations

We consider now $GL(2,\mathbb{C})$, the *General Lineal Group* consisting of the 2x2 matrices

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{2.4}$$

with coefficients in $\mathbb{C}$ and determinant different from zero.

We may simply see that map θ,

$$\begin{aligned} \theta : GL(2,\mathbb{C}) &\to Aut(\hat{\mathbb{C}}) \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\to T(z) = \frac{az+b}{cz+d}, \end{aligned} \tag{2.5}$$

is a homomorphism. If $M,N \in GL(2,\mathbb{C})$ and U,T are Möbius transformations such that $\theta(N) = U$ and $\theta(M) = T$, then $\theta(NM) = U \circ T(z) = \theta(N)\theta(M)$, the definition of a homomorphism.

The kernel of θ can be also very simply determined. Taking into account the definition of the kernel,

$$ker(\theta) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\mathbb{C}) : \theta\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = id_{\hat{\mathbb{C}}} \right\}. \tag{2.6}$$

$id_{\hat{\mathbb{C}}}$ is determined by $T(z) = \frac{\lambda z}{\lambda} = z$. Its matrix representation is of the form $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$, which fulfills that the direction of the projection line is not altered.

By the First Isomorphism theorem of groups (e.g. Bujalance-Garcia et al. (2003)) because the map θ, equation (2.5), is surjective, $GL(2,\mathbb{C})/Ker(\theta) \cong Aut(\mathbb{S}^2)$. The last construction, $GL(2,\mathbb{C})/Ker(\theta)$, is exactly the definition of the *Projective General Linear Group*, $PGL(2,\mathbb{C})$.

For our purposes it is still more convenient to work with the *normalized* version of $GL(2,\mathbb{C})$, the special linear group $SL(2,\mathbb{C})$, which is the group of all matrices with determinant 1. Applying an analogous procedure to $PGL(2,\mathbb{C})$, we may define the *Projective Special Linear Group*, $PSL(2,\mathbb{C})$ as a quotient set of $SL(2,\mathbb{C})/\mathrm{Ker}(SL(2,\mathbb{C}))$.

We may also study the relationship between the Projective Special Linear Group and the Projective General Linear Group. Clearly, for all $N \in GL(2,\mathbb{C})$, there exists $M \in SL(2,\mathbb{C})$ such that $N = \det(N)M$. But on the other side, $\theta(M) = \theta(N)$. This means that every $T \in Aut(\mathbb{S}^2)$ is such that $ad - bc = 1$. This last statement implies that the matrix equivalent of T is in $SL(2,\mathbb{C})$. So that $PSL(2,\mathbb{C}) = PGL(2,\mathbb{C})$.

2.1.3 Classification of Möbius Transformations

Given two transformations, f and g, we say they are conjugated if there exists a third one, u, such that it is verified that $g = u \circ f \circ u^{-1}$.

The Möbius transformations have been historically classified into three or four categories, see Jones and Singerman (1987) or Marden (2007).

The classification followed relies on the trace of the associated matrices to the transformations. Also, it considers the *standard forms* to which Möbius transformations can be conjugated. Following Beardon (1983), those *standard forms* are either the transformation

$$T(z) = kz, \;\; k \neq 1, \tag{2.7}$$

or the transformation

$$T(z) = z + 1. \tag{2.8}$$

If A is a Möbius transformation, then A can be included in one of the following categories:

- **Parabolic:** A is conjugate to $z \to z+1$, a translation. $A \neq id$. $tr(A) = \pm 2$.
- **Elliptic:** A is conjugate to $z \to e^{2i\theta}z$, $\theta \neq \pi$, a rotation. $|A| = 1$. $tr(A) \in (-2,2)$.
- **Loxodromic:** A is conjugate to $z \to \lambda^2 z$, where $|\lambda| > 1$. $|A| = 1$. $tr(A) \in \mathbb{C} \setminus [-2,2]$.

A fourth category, usually appearing in the textbooks like Jones and Singerman (1987), is:

- **Hyperbolic** is a loxodromic transformation whose trace is real. A is conjugate to $z \to \lambda^2 z$ and $\lambda > 1$.

Möbius transformations have a characteristic of the uttermost importance for our work. Möbius transformations have the so-called *fixed points*. Those are the solution, or solutions, of $A(z) = z \; A \neq id_{\hat{\mathbb{C}}}$, that is, of $cz^2 + (d-a)z - b = 0$.

For example, the transformation $z \to z+1$ keeps one and only one point fixed, the ∞. This notion can be extended to the rest of transformations associated to our classification. We find out that for the categories described above we have:

- **Parabolic:** A has exactly one fixed point in $\mathbb{S}^2$.
- **Elliptic:** A has exactly two fixed point in $\mathbb{S}^2$.
- **Loxodromic:** A has exactly two fixed point in $\mathbb{S}^2$.

The intuitive meaning of the fixed points of a loxodromic transformation can be better understood observing Figure 2.1. In this picture the successive images of the puppet obtained by consecutive application of a loxodromic transformation are represented. The consecutive transformations create a kind of *stroll* for the puppet. The fixed points of the transformation appear to be as the apparent *sinks* and *sources* of the spirals bounding the paths of the puppet.

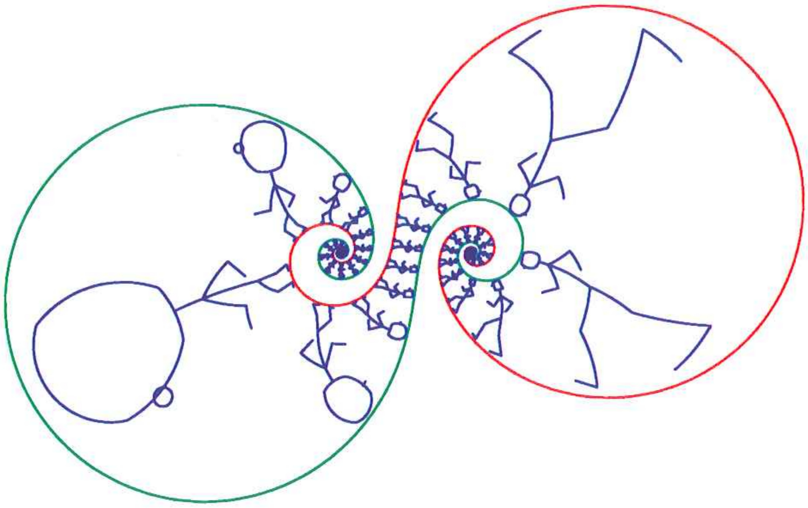


Figure 2.1: Successive transformations of a puppet obtained applying consecutive loxodromic transformations. Taken from Mumford et al. (2002).

Figure 2.1 also helps to introduce a related topic. The fact that one of the fixed points is repulsive and the other attractive. Clearly, in Figure 2.1 the puppet *walks* from the fixed point on the left to the one on the right.

Let us characterize this intuitive concept more mathematically. Suppose that the fixed points of a Möbius transformation f are distinct. f is then conjugated to a transformation of the type represented by the dilation of equation (2.7). Let consider that the fixed points are located at x_1, x_2. The map $g(z) = (z - x_1)/(z - x_2)$ transforms x_1, x_2 into $0, \infty$. Thus, $g \circ f \circ g^{-1}(z) = kz$. Operating we obtain,

$$\frac{f(z) - x_1}{f(z) - x_2} = k\frac{z - x_1}{z - x_2}. \tag{2.9}$$

Calculating the derivatives at x_1, it is immediate to obtain that $f'(x_1) = k$. By an analogous treatment substituting x_1 by x_2 in g we obtain $f'(x_2) = 1/k$.

Assuming that $|k| > 1$, x_1 is called the repulsive fixed point and x_2 is the attractive fixed point. By the use of the derivatives, the notions of approaching and distancing become explicit. The attractive fixed point is usually denoted by Fix^+A and the repulsive fixed point is denoted by Fix^-A.

Fixed points can also be understood in terms of limits. Taking $x \in \hat{\mathbb{C}}$, so that $x \neq x_1$ or $n \neq x_2$, we have that $\lim_{n\to\infty} f^n(x) = x_2$ and $\lim_{n\to\infty} f^- n(x) = x_1$. The repelling fixed point can be understood as the attractive fixed point of the inverse.

Fractional reflections also can have fixed points.

Proposition 1. *The set of fixed points of a fractional reflection is either empty, or it contains one point, two points, or a cycle in* $\hat{\mathbb{C}}$.

Proof. We consider the transformation written in the form defined by equation 2.3.

The proof is divided into two parts, segregating the cases in which ∞ is either a fixed point or not.

If ∞ is a fixed point, we have that $c = 0$. We also can assume that $d = 1$. Consequently, we need to solve the equation $z = a\bar{z} + b$. We divide the equation into its real and complex parts considering that $z = x + iy$ and utilizing sub-indexes 1 and 2 for real and complex parts respectively of the rest of the variables. We obtain

$$(a_1 - 1)x + a_2 y = -b_1, \tag{2.10}$$

$$a_2 x - (a_1 + 1)y = -b_2. \tag{2.11}$$

Two linear equations. The solution set of the previous equations is either empty, or a point or a line depending on the value of the parameters. This solution must be combined with the known fixed point at ∞. The fixed set is thus constituted by either a point, two points, or a circle.

We address the second case. If $c \neq 0$, we need to solve the equation

$$|z|^2 + dz - a\bar{z} - b = 0. \tag{2.12}$$

We decompose it into its real and imaginary parts. We obtain two equations,

$$x^2 + y^2 + (d_1 - a_1)x + (a_2 - d_2)y - b_1 = 0, \tag{2.13}$$

$$(d_2 - a_2)x + (d_1 + a_1)y - b_2 = 0. \tag{2.14}$$

Equation (2.13) is the equation of a circle. But its radius must not be positive. Equation (2.14) is linear in two variables. Its solution is either empty, a line, or a plane. □

In general, because of its analogy with *normal life* reflection in mirrors, etc., the Fractional Reflections that have a circle of fixed points are simply referred to as **reflections**. Note that a line is a also a circle, but with an infinite radius.

2.1.4 Commutators

Definition 3. *Given two elements f and g of group G, the element,*

$$[f,g] = f \circ g \circ f^{-1} \circ g^{-1}, \tag{2.15}$$

*is called the **commutator** of f and g*[1].

In this section we are going to be concerned by the relationship of commutators and fixed points. Concretely, we will be interested by the solutions of

$$[f,g] = 1, \tag{2.16}$$

namely, the cases in which f and g commute.

We may slightly alter the equation (2.16) to obtain

$$f \circ g \circ f^{-1} = g. \tag{2.17}$$

From this last equation, it is evident that if f and g commute, f keeps the fixed points of g invariant.

We may also investigate how the fixed points of g correlate with the fixed points of f.

If g is parabolic and has a fixed point in x, interchanging the roles of f and g, then f also has a fixed point in x. This means that two parabolics with the same fixed points commute.

Now consider the case in which g has two fixed points. g can be normalized so that $g(z) = \lambda^2 z$. This fixes both 0 and ∞. Then, either f also fixes these two points or interchanges them. If both f and g fix the same points, then they commute. If f interchanges the fixed points of g then g also interchanges the fixed points of f.

We will finalize this section providing a theorem whose results will be utilized later in this document.

Theorem 2. *If f has exactly two fixed points and f and g share exactly one fixed point, then the commutator is parabolic.*

Maskit (1988). Let us consider that the transformations have been normalized. Thus, f has fixed points in 0 and ∞ and g in ∞. The associated matrices of f and g adopt the shapes,

$$f = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \tag{2.18}$$

and

$$g = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, \tag{2.19}$$

[1] Bujalance-Garcia et al. (2003) defines the commutator as $[f,g] = f^{-1} \circ g^{-1} \circ f \circ g$ while Beardon (1983) utilizes the definition given above.

where $a, b, c, d, t \in \mathbb{C}$. Operating we find out that,

$$[f,g] = \begin{pmatrix} 1 & -ab + t^2ab \\ 0 & 1 \end{pmatrix}. \tag{2.20}$$

But $ab(t^2 - 1) \neq 0$. $[f,g]$ adopts the shape $z \to z + k$ where $k = -ab + t^2ab$. $[f,g]$ is parabolic. □

2.1.5 System of Generators

We continue our study considering some special cases of Möbius transformations. Because of their significance, these cases should be studied particularly. They are:

i) $z \to e^{i\theta}z, \theta \in \mathbb{R}$. These constitute the rotations of the Riemann sphere around the vertical axis

ii) $z \to 1/z$ which represents two fractional reflections, an *inversion* ($z \to 1/\overline{z}$) and a reflection on the real axis ($z \to \overline{z}$).

iii) $z \to rz, r \in \mathbb{R}$, represents a similarity transformation implying an expansion, or a contraction, by a factor r.

iv) $z \to z + 1$ which acts on the plane as a translation.

The significance of the transformations we have summarized above lie in a notable fact.

Theorem 3. *Every Möbius transformation is a composition of finitely many transformations of types i), ii), iii), iv).*

Proof. Jones and Singerman, 1987. We consider the typical representation of the Möbius transformations in the form $T(z) = (az + b)/(cz + d)$ with $ad - bc = 1$.

1. If $c = 0$, $T(z) = (az + b)/d$ with $a, d \neq 0$. We may write $a/d = re^{i\theta}$ and $b/d = t$ so that $T(z) = re^{i\theta} + t$. We decompose T. We rewrite it as $T(z) = T_t \circ S_r \circ R_\theta$ with $R_\theta(z) = e^{i\theta}z$, $S_r(z) = rz$ and $T_t(z) = z + t$. We finished for $c = 0$.

2. Now we study the case in which $c \neq 0$. Dividing, we obtain

$$\begin{aligned}(az + b)/(cz + d) &= a/c + (bc - ad)/(c[cz + d]) = \\ &= a/c - 1/(c[cz + d]).\end{aligned} \tag{2.21}$$

   We rewrite last expression as,

$$(az + b)/(cz + d) = T_t \circ J(-c^2z - cd), \tag{2.22}$$

   where $J(z) = 1/z$. And so,

$$(az + b)/(cz + d) = T_t \circ J \circ T_{-cd} \circ S_{-c^2}. \tag{2.23}$$

□

CHAPTER 3

KLEINIAN GROUPS

In this chapter, we define the Kleinian Groups and some other related fundamental concepts necessary for the ulterior analysis of the limit set. With this goal, we characterize the Kleinian Groups as discontinuous discrete groups. On this basis we study the convergence of a sequence of distinct elements of a Kleinian group.

3.1 Discontinuous groups

3.1.1 General

Let X be a topological space and G a group of self-homeomorphisms acting on X.

Definition 4 (Freely discontinuous)**.** *The action of G on a point $x \in X$ is freely discontinuous (Maskit (1988)) if there is a neighborhood of x, U, so that $g(U) \cap U = \emptyset$ for all $g \in G$ non-trivial. If the neighborhood U verifies the previous property it is called a nice neighborhood.*

The freely discontinuous notion is simple. It establishes that all translations of U do not overlap.

Definition 5 (Free regular set)**.** *The set of points in X at which the action of G is freely discontinuous is called the Free Regular Set. We denote it by $\Omega^o(G)$ following the nomenclature of Maskit (1988).*

For simplicity, we will denote $\Omega^o(G)$ simply as Ω^o when no confusion can arise. Following Series (2005), Ω^o can receive several alternative denominations. It can be called *Regular Set* or *Domain of Discontinuity*.

Definition 6 (Kleinian Group)**.** *A subgroup G of Möbius transformations whose action is freely discontinuous in some point $x \in \hat{\mathbb{C}}$ is called Kleinian group.*

With its simplicity, this definition is key for this work and will be utilized extensively. It is therefore advisable to have it very well present.

3.1.2 A Small Digression on the Introductory Bibliography on Kleinian Groups

Kleinian groups have received considerable attention since the seminal work of Fricke and Klein (1897).

At this moment, it is pertinent to comment superficially on the introductory literature available on the topic of Kleinian Groups.

After having carried out a review of the literature, the author considers the basic introductory works due to Marden (1974) and Series (2005) as an excellent first step for any newcomer in the topic. Both works provide a very concise global overview on the topic.

As a second step, the newcomer may consider the work of Mumford et al. (2002). This constitutes a recreational fresh publication, that in spite of its simplicity may help to fix some concepts and generate enthusiasm.

Further insight on the topic can be achieved through the superb works of Maskit (1988) and Marden (2007). The targeted audience of the two previous books is postgraduates. Therefore, they contain many concepts of some advanced level that are considered to be known. Thus, the works of Ahlfors (1953) and Beardon (1983) are a great complement to the texts of Maskit (1988) and Marden (2007).

3.2 Discrete Groups

3.2.1 General

We start this section making a small digression into Topology. As such we will restate a couple of definitions without intending to be extensive. More details on the topic can be found in Arregui Fernandez (1998).

A **cover** of a particular set is a collection of sets whose union contains the set as a subset. An **open cover** is a cover in which all sets are open. A **discrete subgroup**, $H \subset G$, is one such that there exists an open cover of H in which every open subset contains exactly one element of H.

With these considerations, a **discrete topology** can be defined creating an open subset for all members of the topological space. Inside of the discrete topology, the members of the set form a discontinuous sequence in which they are virtually isolated from each other. This is the notion of **discreteness** in which we are interested.

The Möbius transformations inside of the Kleinian groups are an example of a discrete subgroup. We will prove this statement in Proposition 3. We may define its *natural topology* considering the matrices equivalent to the members to the Möbius transformations.

Definition 7 (Convergence of Möbius transformations)**.** *We may say that a certain sequence* $\{g_m\}$ *of members of group* G *converges to a certain transformation* $g \in G$ *if each entry of the equivalent matrix of* g_m *converges, as a complex number, to the corresponding entry of* g.

We give now two important properties of convergence. They also illustrate how discreteness and convergence are related.

Proposition 2. *If* G *is a non-discrete sub-group of the Möbius transformations. There is a sub-sequence of distinct elements of* G *converging to the identity.*

Proof. If G is non-discrete, there is a sequence $\{g_m\}$ of elements of G converging to some Möbius transformation g. We may consider that the sub-group G has been normalized. Therefore we may express any g either as $g(z) = z + 1$ or $g(z) = K^2 z$ for some K. The composition of $g_{m+1} \circ g_m^{-1}$ converges to the identity. □

Proposition 3. *A Kleinian group is a discrete subgroup of the Möbius transformations.*

Proof. We utilize Proposition 2, supposing that the Kleinian group is not discrete. Thus, there is a sequence of $\{g_m\}$ such that $g_m \to 1$ and therefore, $g_m(z) \to z$. This means that in any neighborhood of z there are infinitely many translations of z. Thus $z \notin \Omega^0(G)$. □

3.2.2 Normalization, Cross Ratio

It is possible–this we will see immediately–to have the images of up to three points conveniently located through a transformation.

Concretely, G is a Kleinian Group, g a transformation in G and h is a Möbius transformation. The points, $z_1, z_2, z_3 \in \hat{\mathbb{C}}$ can be mapped into $x_1, x_2, x_3 \in \hat{\mathbb{C}}$ through h. Obviously, this is of interest due to the conjugation $h \circ g \circ h^{-1}$. If we apply the transformation h to all members of G, carrying out the conjugation of the whole group, hGh^{-1}, we say that group G has been *normalized*. The convenient locations of the points x_1, x_2, x_3 are usually 0, 1 and ∞.

A simple way to make the normalization is by the use of the cross ratio.

Definition 8 (Cross ratio)**.** *The cross ratio is the transformation,*

$$z \to \frac{(z - p_3)(p_2 - p_4)}{(z - p_4)(p_2 - p_3)},$$

which has the property of being the unique Möbius transformation which transforms p_2 *to 1,* p_3 *to 0 and* p_4 *to* ∞*. The cross ratio is expressed with the tuple* (z, p_2, p_3, p_4).

3.2.3 Jorgensen Inequality

Based on the previous results, we may endeavor to approach to an important result for discrete sub-groups, the Jorgensen Inequality, Theorem 5.

Firstly, we carry out the proof of theorem 4. We will utilize this result in the proof of the Jorgensen Inequality and in other sections inside of this document.

Theorem 4. *Let f and g be non-trivial Möbius transformations. f is loxodromic and f and g have exactly one fixed point in common. The group generated by f and g, $< f, g >$ is not discrete.*

Proof. We can assume that $< f, g >$ is normalized. Thus, we consider the common fixed point located at ∞. Also, that f has its second fixed point (repulsive) in 0. We may need to utilize f^{-1} instead of f to flip the attractive and repulsive fixed points if necessary.

Further, based in Theorem 2, we can assume that g is parabolic. The pertinence of this assumption can be easily confirmed later observing the form of equation (3.1) and comparing it with Theorem 2.

We take the matrix forms of the transforms, $f = \begin{pmatrix} k & 0 \\ 0 & k^{-1} \end{pmatrix}$ with $|k| > 1$, and $g = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$.

Computing,

$$f^{-m} \circ g \circ f^{m} = \begin{pmatrix} 1 & bk^{-2m} \\ 0 & 1 \end{pmatrix}. \tag{3.1}$$

This tends to I. Applying proposition 2, this means that $< f, g >$ is not discrete. □

Theorem 5 (Jorgensen Inequality)**.** *The Jorgensen Inequality establishes that if: i) the Möbius transformations f and g generate a discrete subgroup (like a Kleinian one), ii) f is loxodromic , iii) f and g do not share a common fixed point iv) g does not keep the fixed point of f invariant; then it is verified that*

$$|tr^2(f) - 4| + |tr([f, g]) - 2| \geq 1. \tag{3.2}$$

The clause ii) is not strictly imperative. Jorgensen Inequality can be extended to the cases in which f is parabolic or elliptic. Nevertheless, last cases result in a more complex proof. Therefore, we just discuss loxodromic case here.

Proof. We consider the normalized forms of f and g,

$$f = \begin{pmatrix} k & 0 \\ 0 & k^{-1} \end{pmatrix}, \qquad |k| > 1, \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{3.3}$$

We start justifying that a, b, c and d are non-zero.

- If $a = 0$, $g(\infty) = 0$. The conjugate, $g \circ f \circ g^{-1}$, and f have the same fixed points. This means that $g(0) = \infty$ implying that also $d = 0$. But this is in contradiction to the hypothesis. The same argument applies to the hypothesis $d = 0$.

- $b = 0$, or $c = 0$, implies that f and g share a common fixed point. This is also in disagreement with the hypothesis of the theorem.

We will now recursively define the sequence $\{g_n\}$. As initial value we set $g_0 = g$. We define recursively,

$$g_{m+1} = g_m \circ f \circ g_m^{-1}. \tag{3.4}$$

We will write down g_m as,

$$g_m = \begin{pmatrix} a_m & b_m \\ c_m & d_m \end{pmatrix}. \tag{3.5}$$

Expanding and operating equation (3.4), it is found out that,

$$a_{m+1} = a_m d_m k - b_m c_m k^{-1}, \tag{3.6}$$

$$b_{m+1} = a_m b_m (k^{-1} - k), \tag{3.7}$$

$$c_{m+1} = c_m d_m (k - k^{-1}), \tag{3.8}$$

$$d_{m+1} = a_m d_m k^{-1} - b_m c_m k. \tag{3.9}$$

At this stage, we calculate a couple of intermediary results. These correspond to the expansion of the terms of the Jorgensen Inequality, equation (3.2) in terms of the variables defined above. Thus,

$$\left|tr^2(f) - 4\right| = \left|(k + k^{-1})^2 - 4\right| = \left|k + k^{-1}\right|^2, \tag{3.10}$$

and

$$\begin{aligned} |tr[f,g] - 2| &= \left|2ad - (k^2 + k^{-2})bc - 2\right| = \\ &= \left|2 + 2bc - (k^2 + k^{-2})bc - 2\right| = |bc| \left|k - k^{-1}\right|. \end{aligned} \tag{3.11}$$

We combine them and set the variable α

$$\alpha = \left|tr^2(f) - 4\right| + |tr[f,g] - 2| = (1 + |bc|)\left|k - k^{-1}\right|^2. \tag{3.12}$$

We formulate the assumption of the theorem. We set that

$$\alpha < 1, \tag{3.13}$$

that is, the inverse of the Jorgensen Inequality is verified. We will continue our rationale considering the assumption to find a contradiction and prove the theorem.

We will devote the rest of this proof to the study of the behavior of the variables a_m, b_m, c_m and d_m. If they all converge, group G will stop being discrete and we would have arrived at a contradiction.

Let us prove that a_m, b_m, c_m and d_m are all different from 0.

The case $m = 0$ has already been treated in the rationale above.

Let us consider the case in which $m > 0$.

Below equation (3.3) we have shown that $a = 0 \Leftrightarrow d = 0$. By the same rationale, also must happen that $a_{m+1} = 0 \Leftrightarrow d_{m+1} = 0$.

$a_{m+1} = 0$ and $d_{m+1} = 0$ imply that $k^2 = (b_m c_m)/(a_m d_m)$ and $k^2 = (a_m d_m)/(b_m c_m)$. Thus, $k^4 = 1$. But f is loxodromic, so that this is excluded.

From equation (3.7), due to the induction hypothesis we see that if $a_m \neq 0$ then also $b_{m+1} \neq 0$. Analogously for c_{m+1}.

We have managed to prove that a_m, b_m, c_m and d_m are all different from 0.

Let us study the convergence of a_m, b_m, c_m and d_m. We rewrite equations (3.6) and (3.9) to obtain,

$$a_{m+1} = (1 + b_m c_m)k - b_m c_m k^{-1} = k + b_m c_m (k - k^{-1}) \tag{3.14}$$

$$d_{m+1} = (1 + b_m c_m)k^{-1} - b_m c_m k = k^{-1} + b_m c_m (-k + k^{-1}) \tag{3.15}$$

We have written a_{m+1} and d_{m+1} as a function of $b_m c_m$. Thus, we need to study the behavior of $b_m c_m$. Let us analyze it,

$$\begin{aligned} |b_{m+1} c_{m+1}| &= |a_m b_m c_m d_m| \left|k - k^{-1}\right|^2 = \\ &= |b_m c_m| |1 + b_m c_m| \left|k - k^{-1}\right|^2 \leq \\ &\leq |b_m c_m| (1 + |b_m c_m|) \left|k - k^{-1}\right|^2 \end{aligned} \tag{3.16}$$

For $m = 0$,

$$|b_1 c_1| \leq (1 + |bc|)|bc| \left|k - k^{-1}\right|^2 = \alpha |bc|. \tag{3.17}$$

For $m = 1$,

$$|b_2 c_2| \leq (1 + |b_1 c_1|)|b_1 c_1| \left|k - k^{-1}\right|^2 \leq \tag{3.18}$$

$$\leq (1 + \alpha |bc|)\alpha |bc| \left|k - k^{-1}\right|^2 = \alpha^2 |bc| \tag{3.19}$$

We come back now to equation (3.16) with $|b_{m+1} c_{m+1}| \leq \alpha^m |bc|$. We obtain

$$\begin{aligned} |b_m c_m| (1 + |b_m c_m|) \left|k - k^{-1}\right|^2 &\leq \alpha^m |bc| \left(1 + \alpha^m |bc|\right) \left|k - k^{-1}\right|^2 \leq \\ &\leq \alpha^m |bc| (1 + |bc|) \left|k - k^{-1}\right|^2 \leq \\ &\leq \alpha^{m+1} |bc|. \end{aligned} \tag{3.20}$$

Thus,

$$|b_{m+1} c_{m+1}| \leq \alpha^{m+1} |bc|. \tag{3.21}$$

α is always larger than 0. Due to the fact that $\alpha < 1$, we see that $|b_m c_m| \to 0$. This means also that,

$$a_{m+1} \to k, \tag{3.22}$$

$$d_{m+1} \to k^{-1}. \tag{3.23}$$

We have finished the analysis of a_m and d_m. We now need to study the convergence of b_m and c_m. This will be achieved utilizing the following lemma.

Lemma 1. *Suppose that f and g are elements of G. Consider these terms to be written in the notation of equation* (3.3). *a, b, c and d are all different than zero. It is also verified that $|bc| \leq |k|^2$. Under these circumstances, there is an integer m so that*

$$g' = f^m \circ g \circ f^{-m} = \begin{pmatrix} a' & b' \\ c' & d \end{pmatrix}, \tag{3.24}$$

where $a' = a$, $|b'| \leq |k|^2$, $|c'| \leq |k|^2$ and $d' = d$.

Proof. We operate $f^m \circ g \circ f^{-m}$ to obtain

$$a' = a, \tag{3.25}$$

$$b' = bk^{2m}, \tag{3.26}$$

$$c' = ck^{-2m}, \tag{3.27}$$

$$d' = d. \tag{3.28}$$

We firstly assume that $|c| > |k|^2$. In these circumstances, we choose $m \in \mathbb{Z}$ that verifies $|k|^{2m} \leq |c| \leq |k|^{2m+2}$. Then $|c'| = |ck^{-2m}| \leq |k|^2$ and $|b'| = |bk^{2m}| \leq |k|^{2m+2}/|c| \leq |k|^2$. An analogous treatment is carried out in the case in which $|b| > |k|^2$ which, due to $|bc| \leq |k|^2$, also corresponds to $|c| < |k|^2$. □

We finish now the proof of the Jorgensen theorem. Firstly, observing equation (3.14), we can establish that there is a sequence of $\{g_m\}$ where all $\{a_m\}$ are different. We may now select m large enough so that it is verified that $|b_m c_m| < |k|^2$. We consider the element $g'_m = f^m \circ g_m \circ f^{-m}$ obtained from the application of equation (3.14). It is verified that $a'_m = a_m$, $d'_m = d_m$. Also that $b'_m = b_m k^{2m}$ and $c'_m = c_m k^{-2m}$ which together with equations (3.7) and (3.8) imply that all members of g'_m are distinct.

We remind now that a family of functions h_i from an arbitrary set X in $\mathbb{C}$ is uniformly bounded if there exists a real number M such that $|h_i(x)| \leq M$ for all indexes i and for all points x in X.

The entries of matrix g'_m are uniformly bounded, what means that a convergent sub-sequence can be selected. This is the contradiction we were seeking for, that allows us to prove the theorem. □

3.3 Convergence

We start this section making a small digression. In this document, we will in general be studying properties of the Kleinian groups that do not change with conjugation (in this sense see also section 2.1.4).

3.3.1 Isometric Circle

We may turn our attention to the kernel of this section, the convergence. With this goal we will utilize here the definition of convergence given in section 3.2.1. In spite of the fact that we treat convergence more extensively here, it was necessary to advance the definition because of the strong connections between convergence and discreteness a topic that was analyzed in sections 3.2.1 and 3.2.3.

We begin with the definition of some properties of the Möbius transformations. For our next definition, we assume that $g(\infty) \neq \infty$ or equivalently that $c \neq 0$.

Definition 9 (Isometric Circle)**.** *The isometric circle of a Möbius transformation $T(z)$ is the set*

$$I(T) = \left\{ z \in \mathbb{C} : \mid T'(z) \mid = 1 \right\}.$$

Because

$$T'(z) = \frac{1}{(cz+d)^2},$$

it happens that,

$$\mid T'(z) \mid = 1 \Rightarrow \mid cz + d \mid^{-2} = 1, \; c \neq 0$$

The center of the isometric circle, I, coincides with the point $-d/c$, which corresponds to $T^{-1}(\infty)$ and its radius is $\mid 1/c \mid$. For simplicity, we denote the center of I by $\alpha = T^{-1}(\infty)$.

Similarly, we denote with $\alpha' = T(\infty) = a/c$ the center of the isometric circle, I', of the inverse transformation T^{-1}. The radius of the isometric circles of both T and T^{-1} coincide.

We can give a simple, yet interesting, geometrical interpretation of the previous paragraphs. The family of circles passing through ∞ and α is mapped by T onto the family of circles passing through ∞ and α'. The orthogonal trajectories to the first family are mapped by g to the orthogonal trajectories of the second. Therefore, the family of circles centered at α are mapped onto the family of circles centered at α'. A unique circle I of the first family is mapped onto a circle of the same size. Its image is I'.

3.3.2 Decomposition in reflections

Based on our discussion on isometric circles we will attempt to analyze an argument that appears quite often in the analysis of Kleinian groups. That is the decomposition of any Möbius transformation g into three maps: a reflection in its isometric circle, p, a reflection in the bisector of the segment α to α', q, and a turning with *vertical* axis passing through α'. Namely,

$$g = r \circ q \circ p \tag{3.29}$$

see Figure 3.1.

Fixing p and q as reflections in a circle and a line as described above, and establishing the decomposition of equation (3.29), we need to investigate the nature of r. That is, we assume the decomposition of equation (3.29) fixing p and q for a known g to investigate r.

We will write down $r = g \circ (q \circ p)^{-1}$ and explore its behavior. It is significant to note that,

$$r^{-1}(\infty) = q \circ p \circ g^{-1}(\infty) = q \circ p(\alpha) = q(\infty) = \infty. \tag{3.30}$$

The second equality comes from the definition of α. The third corresponds to the circle reflection of the center of the circle. Last equality, $q(\infty) = \infty$, represents a reflection in a straight line of a point in ∞.

We also must note that,

$$r^{-1}(\alpha') = q \circ p \circ g^{-1}(\alpha') = q \circ p \circ g^{-1}(g(\infty)) = q \circ p(\infty) = q(\alpha) = \alpha'. \tag{3.31}$$

Those equations are pretty similar to the last ones. $p(\infty) = \alpha$ is the circle reflection utilizing the isometric circle of a point in ∞. Clearly it corresponds to the center. Last equality, the reflection in the straight line simply flip α onto α'.

r preserves the points ∞ and α', the center of I'. This indicates that we look for a transformation of the form $k^2(z - \alpha') + \alpha'$ or $k^2(\bar{z} - \bar{\alpha}') + \alpha'$. Both verify previous conditions. We may now study how r transforms the circle I'.

$$r^{-1}(I') = q \circ p \circ g^{-1}(I') = q \circ p(I) = q(I) = I'. \tag{3.32}$$

Thus r also preserves I'. Therefore, we conclude that r may adopt the two previously mentioned shapes,

$$r(z) = k^2(z - \alpha') + \alpha', \tag{3.33}$$

or

$$r(z) = k^2(\bar{z} - \bar{\alpha}') + \alpha', \tag{3.34}$$

with $|k| = 1$. It will adopt one form or the other, depending on whether r preserves or reverses the orientation. In the decomposition we are considering, it is also important to realize that r and q are isometries in the euclidean plane but p is not.

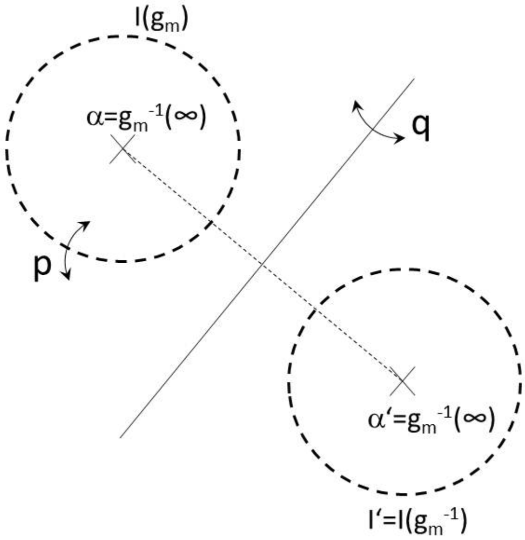


Figure 3.1: Isometric circles and sketch of the transformations p and q.

3.3.3 Convergence of the Radius of the Isometric Circles

At this stage, we may consider a distance in the sphere. It does not matter whether we are taking into account the chordal distance or the distance over the surface. Based on this chosen distance, we may define the spherical diameter of a set, as the maximum of the distances between its distinct points. We denote this spherical diameter of a set X by $dia(X)$. Based on the diameter, we may consider the area of the set making an analogy with the surface of the circle. We denote this area with $meas(X)$. Note that the spherical metric and the euclidean are equivalent. Stereographic projection, but near ∞, has a bounded distorsion of distances. Thus, given a bounded set there is a constant $K > 0$ so that for all $x, y \in U$,

$$K^{-1}|x-y| \leq d(x,y) \leq K|x-y|, \tag{3.35}$$

where $d(\cdot,\cdot)$ represents the spherical distance.

We will study the evolution of the radius of the isometric circles of successive transformations. Note that the radius of isometric circles is $\rho = 1/|c|$. In order to obtain some conclusions, we will consider a sequence of transformations. Then, we will study the sum of the radius of the isometric circles. We will see that:

Theorem 6. *If ∞ is in Ω^0 then,*

$$\sum{}' |c|^{-4} < \infty, \tag{3.36}$$

where $\sum'$ denotes summation over all non-trivial elements of G.

Proof. We consider a nice neighborhood, U, of ∞. U has the form of $\{z : |z| > \rho\} \cup \infty$. Let α be the center of the isometric circle I of a non-trivial element g of G. U is nice, which means that the center of the isometric circle $\alpha = g^{-1}(\infty) \notin U$. Let δ be the euclidean distance from α to U. Clearly $\delta \leq \rho$.

Now, let us consider the reflection $p(z)$ in the circle $|z| = \rho$. The reflection has the form $p(z) = \rho^2/\bar{z} = \rho^2 z/|z|^2$. z and $p(z)$ lie in the same radius from the origin. The locations of the original z and the image $p(z)$ are related as $|p(z)| = \rho^2/|z|$. If we consider the reflection in the isometric circle, I, whose center is α, z and $p(z)$ lie in the same radius emanating from α. The relative locations of the point and its image is $|p(z) - \alpha| = \rho^2/|z - \alpha|$.

We consider now the point x of U which is closer to α. Then $\delta = |x - \alpha|$. $p(U)$ lies inside of the circle of radius ρ^2/δ. But $\alpha = g^{-1}(\infty)$ lies inside of $p(U)$. Thus,

$$\frac{\rho^2}{\delta} \leq dia(g(U)) \tag{3.37}$$

We taking into account the decomposition (3.29). r and q are isometries. p is the only member of the decomposition that is not an isometry. This means that

$$dia(g(U)) \geq |c|^{-2}\delta^{-1}. \tag{3.38}$$

Because $g(U)$ is a circular disc contained in the complement of U, it is bounded. There is a constant $K > 0$ such that,

$$meas(g(U)) \geq K^{-1} dia^2(g(U)). \tag{3.39}$$

Operating equations (3.38) and (3.39), we obtain,

$$\sum{}' |c|^{-4} \leq \sum{}' \delta^2 dia^2(g(U)) \leq K\rho^2 \sum{}' meas(g(U)). \tag{3.40}$$

Now we consider the last sum. For simplicity, we examine the projections of $g(U)$ into the Riemann Sphere. The distortion of the stereographic projection is bounded except near ∞. The $g(U)$ are disjoint. Thus, the sum of their areas must be finite. Thus,

$$\sum{}' |c|^{-4} \leq \infty. \tag{3.41}$$

□

We may summarize the previous results in the following corollary.

Corollary 1 (Convergenece of the radius of the isometric circles)**.** *Let us consider a sequence $\{g_m\}$ of distinct elements of the Kleinian group G. $\infty \in \Omega^0$ and take ρ_m as the radius of the isometric circle of the transformation g_m. Then, $\rho_m \to 0$.*

3.3.4 Convergence of Möbius Transformation

We may now address the main result of this section related to the convergence of Möbius transformations. To do it, we need some definitions.

Definition 10. *A sequence of functions $\{f_\nu\}$, $f_\nu : X \to \mathbb{C}$,* ***converges uniformly*** *to a limiting function f on an arbitrary non-empty subset $X \subset \mathbb{C}$ if, given any positive number $\epsilon \in \mathbb{R}$, there exists a value $n \in \mathbb{N}$ such that for all $m > n$ each function f_m differs from f no more than ϵ for all points $x \in X$.*

We follow now Freitag and Busam (2006) in the following definitions.

Definition 11. *A sequence of functions $\{f_\nu\}$* ***converges locally uniformly*** *to f if for every point $x \in X$ there is a neighborhood U of x in $\mathbb{C}$ such that $f_\nu|_{U\cap X}$ is uniformly convergent.*

Definition 12. *A sequence of functions is called* ***compactly convergent*** *(also* ***uniformly convergent on compact sets****) if it converges uniformly in any compact $K \subset X$.*

The Heine-Borel theorem establishes that a set $X \in \mathbb{C}$ is compact if and only if it is bounded and closed. Therefore, a locally uniformly convergent sequence of functions is compactly convergent. [1]

Theorem 7 (Convergence of Möbius Transformation)**.** *Suppose $\{T_n\}$ is an infinite sequence of distinct Möbius transformations such that the corresponding fixed points p_n, q_n converge to p, q $\in \mathbb{S}^2$.*

Either $p_n = q_n$ (implying a sequence of parabolic transformations), or T_n is elliptic, or p_n is the repelling fixed point and q_n is the attracting fixed point of T_n (when T_n is loxodromic).

There is a sub-sequence $\{T_n\}$ with one of the following properties:

- *There exists a Möbius transformation T such that $\lim T_k(z) = T(z)$ converging uniformly in $\mathbb{H}^3 \cup \mathbb{S}^2$.*

[1] Remmert (2013) provide an alternative and interesting definition for products (or sums) of functions.

Definition 13. *Let X be a locally compact metric space. f_ν is a continuous function on X with values in $\mathbb{C}$. For a sequence f_ν, the product Πf_ν is called* ***compactly convergent*** *or* ***uniformly convergent on compact sets*** *in X if for every compact set K in X there is an index $m = m(K)$, dependent on K, such that the sequence $p_{m,n} = f_m f_{m+1} f_{m+2} \cdots f_n$ with $n \geq m$ converges uniformly on K to a non-vanishing (nullstellenfrei without zeros) function $\hat{f}_m$.*

For each point $x \in X$,

$$f(x) = \prod f_\nu(x) \ \in \mathbb{C} \tag{3.42}$$

We call f the limit of the product, we write $f = \prod f_\nu$, and on K we have $f|_K = f_0|_K \cdot \ldots \cdot f_{m-1}|_K \cdot \hat{f}_m$.

In this document we will make extensive use of the interpretation of a member T_m of a sequence $\{T_n\}$ of distinct Möbius transformations as the composition of other Möbius transformations, τ_j, for example as $T_m = \tau_1 \circ \ldots \circ \tau_m$. In matrix formulation this translates into a product. Therefore, this second definition of compactly convergent maybe of interest in view of the decomposition of each of the members of the sequence. See example 2 for more clarification in this sense.

- $\lim T_k(z) = q$ *for all* $z \neq p$ *converging uniformly on compact sub-sets* $\mathbb{H}^3 \cup (\mathbb{S}^2 \setminus \{p\})$. *Also* $\lim T_k^{-1}(z) = p$ *for all* $z \neq q$ *converging uniformly in compact sub-sets of* $\mathbb{H}^3 \cup (\mathbb{S}^2 \setminus \{q\})$

In the previous theorem, $\mathbb{H}^3$ represents the hyperbolic space. Due to extension matters we cannot give here more details in this topic and refer the readers to specialized literature such as the monograph work of Matsuzaki and Taniguchi (1998) for extensive details in the topic.

Proof. The cases of one or two fixed points are studied separately, namely $p \neq q$ and $p = q$.

- Case $p \neq q$.

  We consider a point $\zeta \in \mathbb{C}$ distinct from p, q, p_n, q_n for all indexes n. The cross ratio $R_n(z) = (z, \zeta, p_n, q_n)$ converges to $R(z) = (z, \zeta, p, q)$ uniformly in the hypothesis of theorem 7 on $\mathbb{S}^2$.

  $S_n(z) = R_n T_n R_n^{-1}(z)$ fixes $0, \infty$ and has the same convergence properties of $\{T_n\}$. We study this $S_n(z)$. For large indexes, $S_n(z) = a_n z$, with $|a_n| \geq 1$. $|a_n|$ can be bounded or unbounded: i) If it is bounded, then S_n converges uniformly to a Möbius Transformation; ii) If it is not, there exists a sub-sequence for which $\lim a_m = \infty$ which means that the subsequence converges uniformly to $S(z) = \infty$ for any given (compact) neighborhood of $z = 0$.

- Case $p = q$.

  Let ζ_1, ζ_2 be two points in $\mathbb{C}$ and $\zeta_1, \zeta_2 \neq q_n$ or q. We consider the cross ratio $R_n(z) = (z, \zeta_1, \zeta_2, q_n)$ and the conjugation, $S_n(z) = R_n T_n R_n^{-1}(z)$, which fixes ∞ and has the same convergence properties of $\{T_n\}$. For enough large indexes, $S_n(z) = a_n z + b_n$. The other fixed point of S_n is $-b_n/(a_n - 1)$. Let us consider now a subsequence of $\{b_m\}$, whose limit can be bounded or unbounded: i) If it is bounded, $\lim b_m = b \neq \infty$. S_n and $\{T_n\}$ have the same convergence properties. Then considering the second fixed point, $\lim a_m = 1$. This means that $\lim S_m = z + b$; ii) If it is unbounded, $\lim b_m = \infty$. S_m is rewritten like

  $$S_m(z) = b_m \left(\frac{(a_m - 1)z}{b_m} + 1 \right) + z.$$

  Since S_n and $\{T_n\}$ have the same convergence properties and the second fixed point is $-b_n/(a_n - 1)$, $\lim (a_n - 1)/b_n = 0$. We obtain $\lim S_m(z) = \infty$. Its inverse is

  $$S_m^{-1}(z) = \frac{b_m}{a_m} \left(\frac{z}{b_m} - 1 \right).$$

  But,

  $$0 = \lim \frac{a_m - 1}{b_m} = \lim \left(\frac{a_m}{b_m} - \frac{1}{b_m} \right). \qquad (3.43)$$

  Also $\lim b_m = 0$ and therefore $\lim \dfrac{a_m}{b_m} = 0$. Thus, $\lim S_m^{-1}(z) = \infty$ for all z.

□

Example 1. *We may now revisit our introduction to bring a little bit of light to the initial photographs 1.3 to 1.5 in pages 3 to 5.*

Definition 14. *Let C_1, C'_1, C_2, C'_2, ..., C_n, C'_n be a set of disjoint circles in $\hat{\mathbb{C}}$. We also take into account the group $G =< g_1, g_2, \ldots, g_n >$, with $n \geq 1$, which members g_i verify that $g_i(C_i) = C'_i$. Each of the transformations g_i has also the peculiarity that the interior of C_i is mapped in the exterior of C'_i. The groups verifying such conditions are called Schottky groups.*

We remind that a group G is free if there is a set $S \subset G$ such that every member of G can be expressed in an unique form as a product of members of S and their inverses.

A classic result of Maskit (1967) states that:

Theorem 8. *A finitely generated Kleinian group G is a Schottky group if and only if G is free, and every element of G other than the identity is loxodromic*

Because of the necessary mathematical apparatus, the proof of this result exceeds the frames of this work and we do not include it here.

Nevertheless, we need to add that the definition of Schottky groups given and the result of Maskit above are too strict. As can be seen in Maskit (1988, page 82) it is also possible to consider parabolic generators for the cases in which the circles considered are tangent. If the tangent point is a fixed point, then the generator is parabolic. We will come back to this topic in our examples.

The Schottky groups are one of the most common examples utilized in the literature to illustrate Keinian groups (Mumford et al. (2002), Marden (2007), Maskit (1988)...).

Let us consider that the initial circle has a radius r and a center P. Following Definition 14, we want to transform this circle to another one located at Q and with a radius s. We can carry out this transformation in three simple successive steps,

$$z_1 = z - P, \tag{3.44}$$

$$z_2 = rs/z_1, \tag{3.45}$$

$$w = z_2 + Q. \tag{3.46}$$

Second step corresponds to four simultaneous operations. It can be divided into three steps. The first is a re-scaling of the circle of radius r to unit radius, $z \to z/r$. The second contains the fractional reflection in a unit circle centered in the origin plus a reflection in the horizontal axis ,$z \to 1/z$ (see also page 13). The third is a re-scaling on the unit circle to a circle of radius s, $z \to sz$.

A group generated in this way, as show by theorem 8, is freely discontinuous and thus a Kleinian group. Clearly, the transformation is not unique. We may divide the second step into $z_{21} = r/z_1$ plus $z_2 = sz_{21}$, and apply for example, a turn $z = e^{i\theta} z_2 1$ in between. Actually, any transformation of the form $w = (uz + v)/(\bar{v}z + \bar{u})$ with $|u|^2 - |v|^2 = 1$ (Mumford et al., 2002) brings a circle to itself. We may apply any transformation of this type and obtain an arbitrary location of a point of our choice.

CHAPTER 4

LIMIT SET

After the considerations of the previous sections, we conduct now the analysis of the limit set of Kleinian groups from a basic and topological point of view. We start defining the concept of limit set providing some examples for an enhanced comprehension. Thereafter, we characterize the limit set showing some of its basic properties. Our analysis is completed with a very basic investigation of the complementary of the limit set.

4.1 General

Let us now endeavor to give a definition of the Limit Set.

Definition 15 (Limit Point)**.** *A point x is a **limit point** for the Kleinian group G, if there exists a point $x_0 \in \Omega^o(G)$, and there is a sequence $\{g_m\}$ fo distinct elements of G such that $g_m(x_0)$ converge to x.*

Definition 16 (Limit Set)**.** *The set of all limit points is called Limit Set and it is denoted by $\Lambda(G)$.*

The Limit Set also receives alternative denominations. Following Series (2005), it is also known as the *Chaotic Set*.

Example 2. *We may continue Example 1 calculating the limit set of a particular set of Schottky circles.*

We select tangent circles of diameter 1 *with centers in positions $\sqrt{2}i, -\sqrt{2}i$, $\sqrt{2}$, $-\sqrt{2}$. This constitutes the simplest possible tangent set in the most common locations.*

We carry out the transformations utilizing equations (3.44) *to* (3.46)*. We may apply turns if necessary to the transformations so that the tangent points of the circles remain in place and unchanged. With the previous explanations in Example 1 this should not have any difficulty.*

We denote the paired circles with C_a*,* C_A*, and* C_b*,* C_B*. We denote with a the transformation that brings* C_a *into* C_A*. We denote with the capital letter* $A = a^{-1}$ *its inverse. We do this analogously for B circles.*

Henceforth, we calculate the so-called words*, transformations of the type* $abbabAba\ldots$ *with all possible non-canceling combinations. Of course, words are formed by collections of letters. We obtain a set of* $4\cdot 3^n$ *transformations where n is the number of letters of the words.*

Once this last task is accomplished, we may apply the transformations to the circles. We profit from the fact that Möbius transformations map circles to circles to obtain the solutions. This is the classic procedure proposed by Fricke and Klein (1897). Like many of the classical methodologies of its time, the procedure has a straightforward graphical interpretation that underlines its elegance.

The operations described in the paragraphs above have been carried out with the help of the numerical code written by the author and shown in Appendix C. We refer the reader to the brief description of this appendix for further details. Either way the code has an unambiguous interpretation.

The result obtained is depicted in Figure 4.1. In the figure, eight levels of transformations, namely words with up to eight letters, have been represented with different colors.

We may note several issues arising from the observation of this picture. Clearly, the limit set is a circle. Also, it is a Jordan curve.

Definition 17. *A Jordan curve is a non-self-intersecting continuous loop on the plane.*

The Regular Set and the Set of Discontinuity[1] *appear as completely separated. The regular set has two separate components.*

The ratio of convergence–namely to a point–of the different circles is very different. This is represented by the disparity of the size of the transformed circles. Close to the tangency fixed points, the rate of decrease of the diameters of the circles is much smaller than in most of the other points of the limit set. These tangency points have been identified by Mumford et al. (2002) as parabolic, whilst the other points where the ratio of convergence is much faster are of conical type. We refer the readers to Nicholls (1989) for the meaning and a detailed description of parabolic and conical points.

We may now introduce some changes in the generators to complicate the problem slightly.

We recall that the generators were created by the simple transformations $z_1 = z - P$*,* $z_2 = rs/z_1$*,* $w = z_2 + Q$ *applied successively, where P and Q are the centers of the original and objective circles, and r and s their respective radii. We will now introduce a turn between them. The transformation can be thus expressed in a very simple manner into five steps,* $z_1 = z - P$*,* $z_2 = r/z_1$*,* $z_3 = e^{\pi i} z_2$*,* $z_4 = sz_3$*,* $w = z_5 + Q$*.*

Figure 4.2 is obtained. The picture is similar to that of Figure 4.1. The major difference lies in the limit set. The latter has been plotted separately in Figure 4.3. The circles plotted are obtained

[1] We will treat this concept in section 4.4

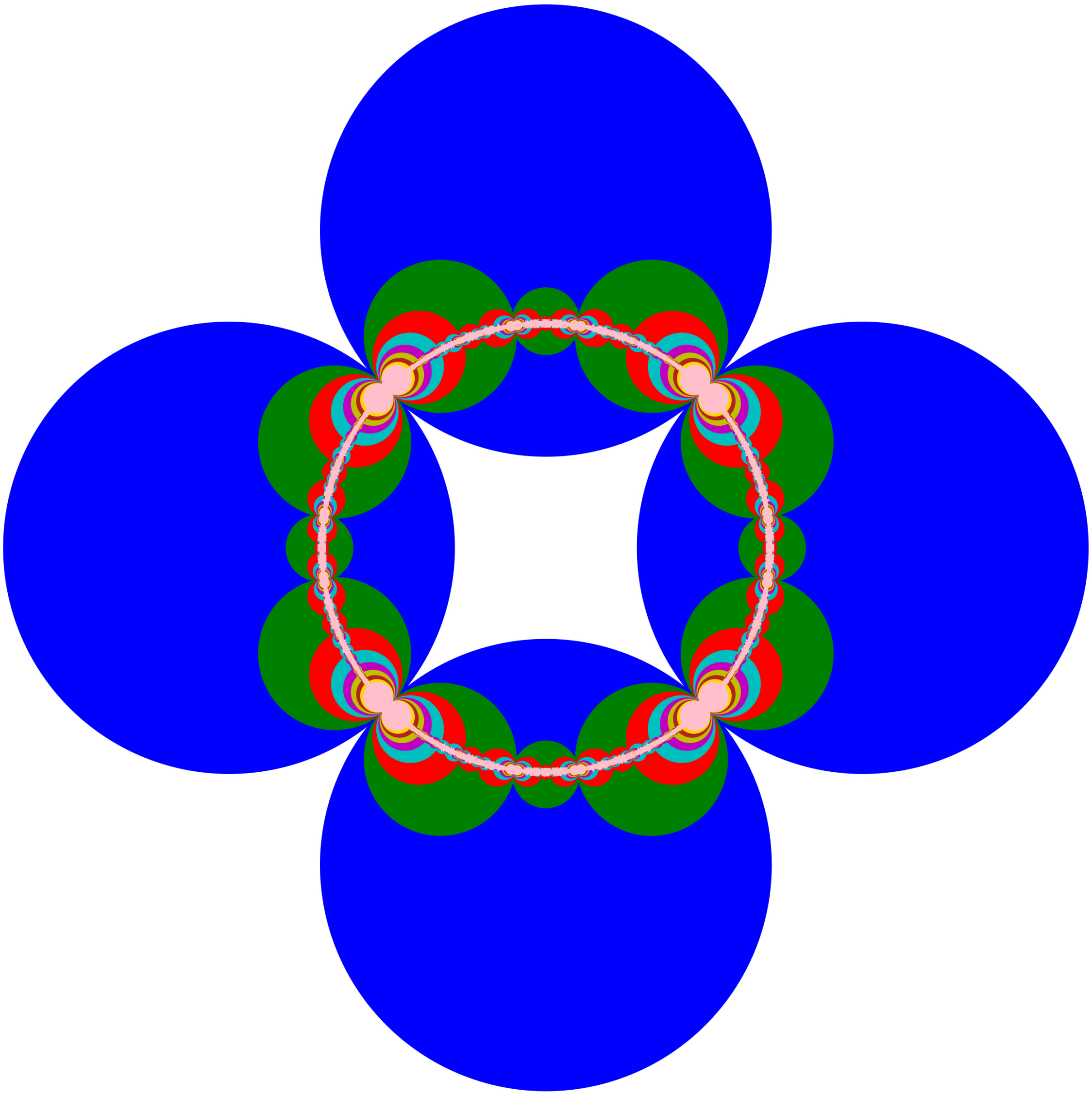


Figure 4.1: Circles obtained after calculations of eight levels of transformations. Words with the same number of letters are depicted with the same color.

after applying eight transformations. They are located in a circle. Nevertheless, they are not tangent to the each other and generate a limit set which is not continuous. By observation of Figure 4.3 it can be stated that the limit set may form a dots and dashes pattern.

We may carry out an additional modification of this example. We may change the original plot of Figure 4.1 avoiding tangency between the circles utilized to calculate the generators. We may also provide initial circles with different radii and avoid symmetry in the figure performing different translations for each circle. The result is the circles depicted in Figure 4.4. Its limit set is sketched separately in Figure 4.5. Clearly the points are no longer located in a circular shape.

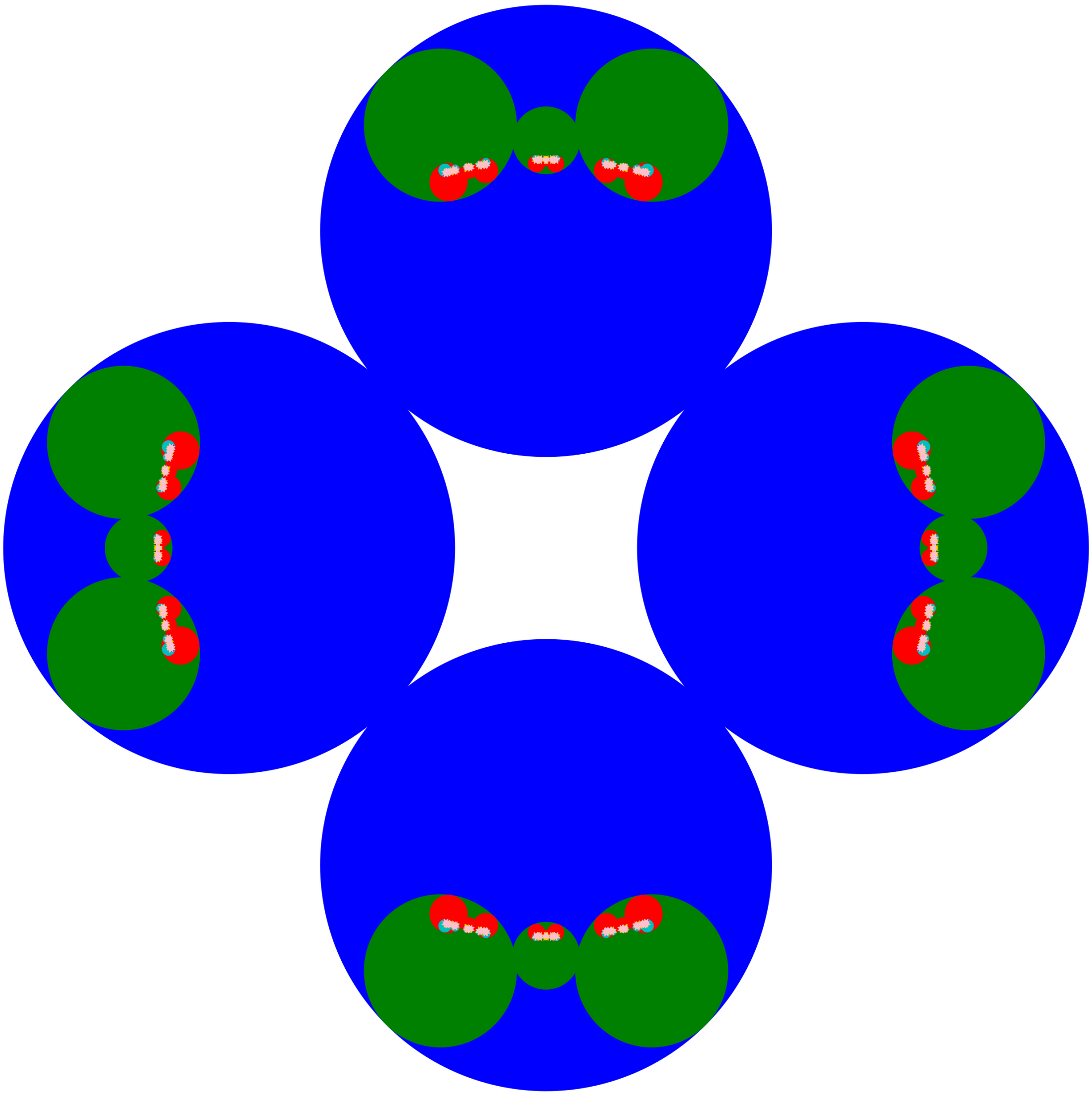


Figure 4.2: Circles obtained after the calculation of eight levels of transformations. The generators were turned (third trasnformation) π rad compared with the ones of Figure 4.1. Words with the same number of letters are depicted with the same color.

They also look more isolated.

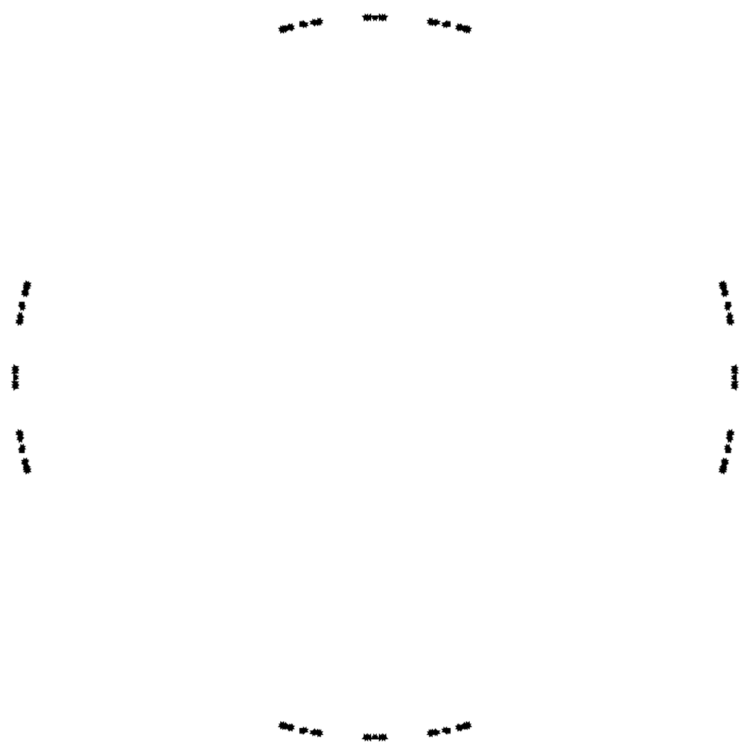


Figure 4.3: Circles obtained after calculation of eight levels of transformations. Only last level is depicted. The generators were turned (third trasnformation) π rad compared with the ones of Figure 4.1.

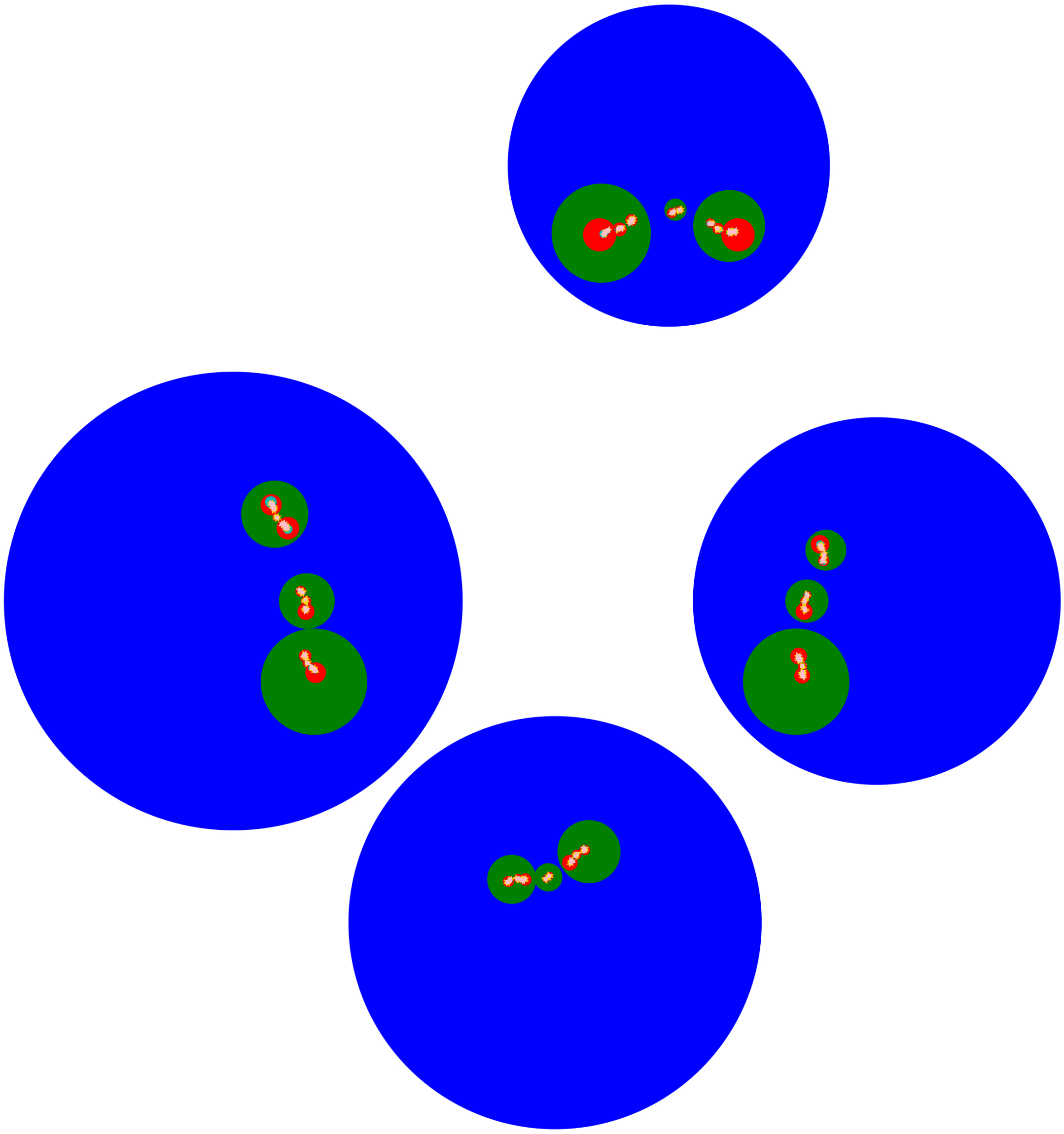


Figure 4.4: Circles obtained after calculations of eight levels of transformations. The generators, through the circles utilized for their calculation, were modified avoiding tangency and symmetry. Words with the same number of letters are depicted with the same color.

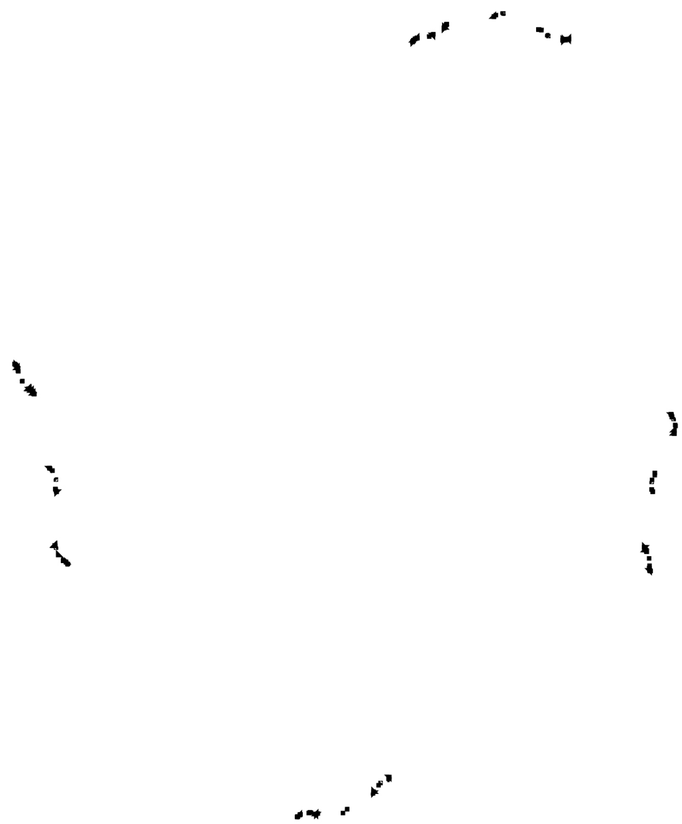


Figure 4.5: Circles obtained after calculations of eight levels of transformations in Figure 4.4. Only last level is depicted.

4.2 Basic Properties, Convergence and Invariance

Let us study forthwith some properties of the limit set. We start with the relationship between the Regular Set and the Limit Set.

Theorem 9. *The intersection of the Free Regular Set and the Limit Set is empty.*

Proof. U is a neighborhood of a $x \in \Omega^0$. Thus, it happens that $g(U) \cap U = \emptyset$. Nevertheless, for any neighborhood, $\mathscr{L}$, of $l \in \Lambda$ there exists a point l_0 and there are infinitely many transforms $g_m \in G$ so that the infinitely many translates $g_m(l_0) \in \mathscr{L}$. This implies that $\Lambda \cap \Omega^0 = \emptyset$. □

Theorem 10. *Let x be a limit point of the Kleinian Group G, $x \in \Lambda(G)$. There is a second limit point y of $\Lambda(G)$, not necessary distinct of x, and a sequence $\{g_m\}$ of distinct elements of G so that $g_m(z) \to x$ converges uniformly on compact subsets of $\hat{\mathbb{C}} \setminus \{y\}$.*

Proof. Because x is a limit point, there is a point z_0 in Ω^0 and there is a sequence $\{g_m\}$ of distinct elements of G so that $g_m(z_0) \to x$.

If required, to fulfill the previous statement we may normalize the functional series to a convenient conjugate.

We may consider that $z_0 = \infty$. We choose a sub-sequence of g_m so that $g_m^{-1}(\infty) \to y$. y is thus a limit point.

We intend to make the decomposition of the map g_m following the format represented by equation (3.29). To accomplish this task, we follow the same procedure and logic of the explanation there, see page 23. Check that section for the definition of isometric circles as well as the logic of this proof.

The center of the isometric circle of g_m, actually the succession of the centers of isometric circles, tends to y (see Figure 4.6).

Let p_m denote the reflection in the isometric circle of g_m. Let q be the reflection in the perpendicular bisector of the line segment between $\alpha = g_m^{-1}(\infty)$ and $\alpha' = g_m(\infty)$. We remind that α and α' are the centers of the isometric circles of the transformation g_m and its inverse. As explained in page 23, if $c \neq 0$ all g_m can be decomposed in terms $g_m = r \circ q \circ p$. q and r are isometries. As it is explained in the rationale devoted to equation (3.29), we take r to be $r(z) = k^2(z - \alpha') + \alpha'$, or $r(z) = k^2(\bar{z} - \bar{\alpha}') + \alpha'$, equations (3.33) or (3.34).

g_m maps the outside of its isometric circle into the inside of the isometric circle of its inverse. It is easy to check this, as $g_m(\infty)$ is the center of $I(g_m^{-1})$ and the image of the center of the isometric circle of g_m, $g_m^{-1}(\infty)$, is ∞. By Corollary 1 the radius of the isometric circles tends to 0. Because $g_m(\infty) \to x$ the center of the isometric circle of g_m^{-1} tends to x.

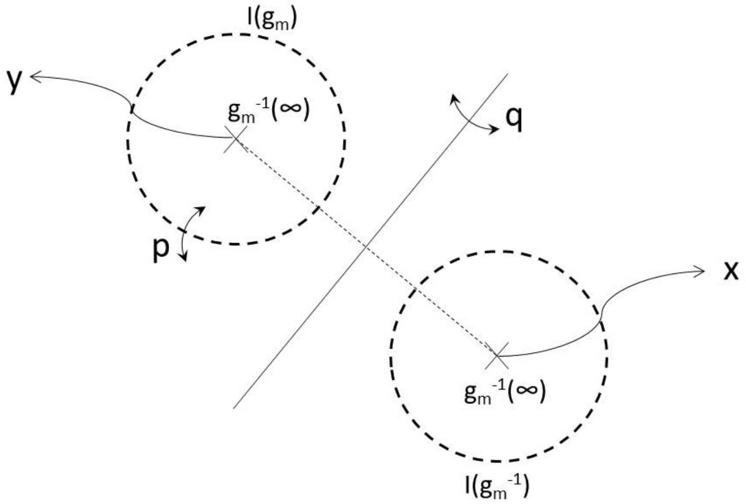


Figure 4.6: Isometric circles of a transformation g_m and its inverse.

By the convergence of $g_m(\infty) \to x$ the uniform convergence of g_m gets proved, as the respective decomposition also converges. □

From the previous theorem, a significant conclusion can be drawn about the nature of the limit set of a Kleinian group: The definition of the limit point depends only on the sequence of the elements of the group and not on the point selected in Ω^0. If $z_0 \in \Omega^0$, so that $\{g_m(z_0)\} \to x$ then $\forall z \in \Omega^0$ there is a sub-sequence of g_m which verifies that $\{g_m(z)\} \to x$.

We turn now our attention to study the invariance of the set.

Definition 18. *A subset Y is G-invariant or invariant under G, if $g(Y) = Y$, for all $g \in G$.*

Theorem 11. *Λ is G-Invariant.*

Proof. Let us consider $x \in \Lambda(G)$ and $g \in G$. Because $x \in \Lambda(G)$, there is a sequence $\{g_m\}$ of distinct elements of G and there is a point z in Ω^0 so that $g_m(z) \to x$. Thus, $g \circ g_m(z) \to g(x)$. This is exactly the definition of the limit point: there is a sequence $\{h_m\}$, which is actually $h_m = g \circ g_m$, that converges to $g(x)$. Therefore, $g(x)$ is in the limit set. □

Example 3. *The results above have a significant importance for the obtaining and plotting of limit sets of Kleinian groups.*

The independence of the limit set on the point or points selected in Ω^0 for its calculation imply that only the computation of the words is significant to determine the limit set.

We may, maybe naively, take a single point, P, and a set of transformations containing as many different words as we may be able to calculate. The points resulting from the application *of the transformations to P should be located at the limit set (actually very close to it). This is also sustained by the fact that Λ is G invariant. Thus, if we obtain a point of the limit set, and we transform it, it will remain in the limit set, contributing to its representation.*

We try now to apply our rationale to a concrete example in order to perform the calculation of a limit set. To be sure that we have a Kleinian group, we will utilize the same procedure as in Example 1 and 2 for the creation of the generators.

In order to obtain a differentiate plot, we will locate the centers of the circles C_1, C'_1, C_2, C'_2 in i, $-i$, $1/2$, $-1/2$. The corresponding radii are $\sqrt{3}/2$, $\sqrt{3}/2$, $\sqrt{2}/2$, $\sqrt{2}/2$ to make them tangent.

The transformation of C_1 into C'_1 and of C_2 into C'_2 utilizing the equations (3.44) to (3.46) immediately provide the generators of the group. As before we denote them as a and b and its inverses as A and B.

Contrary to Example 2, this time there is no need to trace and store the circles. We carry out the calculations in a more abstract way disregarding any graphical interpretation of the algorithm. We simply take the set $\{a(P), b(P), A(P), B(P)\}$ and systematically apply the set of generators to each of the members of the set. To avoid spurious results, a indication of the inverse of the last applied generator is kept. By this simple procedure we avoid considering canceling transformations. For example, taking into account three levels of transformations the procedure results in the set $\{aaa(P), baa(P), \ldots, BBB(P)\}$.

We have automatized all these considerations in the code shown in Annex D. The code was written by own the author in Python language, utilizing Cython (Behnel et al., 2011) to increase numerical efficiency.

Cython allows obtaining a numerically efficient library (a shared object see Hook (2005)) through the pre-compilation of the desired parts of a pseudo-Python code. As its name suggests, it constitutes an intermediate stage between pure Python and C/C++ languages.

To obtain an efficient code, we have written all numerically intensive routines under this modified language. See Annex D for further details of the code.

To obtain the results, we have taken point P located at the origin of coordinates. We have calculated words with ten letters. The results of the calculation are shown in Figure 4.7.

The results converge very fast to the limit set. Actually, very similar results to the ones shown in Figure 4.7 would have been obtained after only five iterations of our procedure, Figure 4.8 (b). Three iterations result nevertheless in insufficient convergence to constitute a significant representation of the limit, Figure 4.8 (a).

We may repeat the procedure carried out in the Experiment 2 in order to see a different convergence behavior.

We introduce therefore a turn between the transformations to obtain one of the generators. We apply this turn only to the generator transforming the horizontal circles in order to seek continuity of the limit set on the points of tangency of the circles utilized for the generators. As explained before in example 1, the transformation is expressed in five steps, $z_1 = z - P$, $z_2 = r/z_1$, $z_3 = e^{\pi i} z_2$,

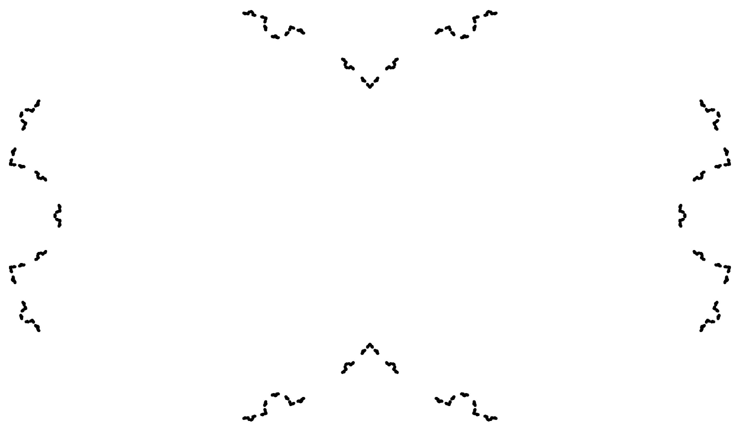
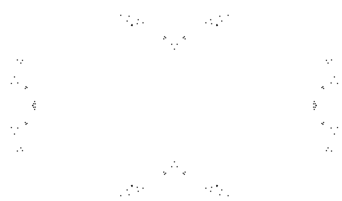
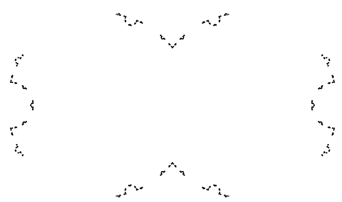


Figure 4.7: Limit set obtained transforming the origin of coordinates.



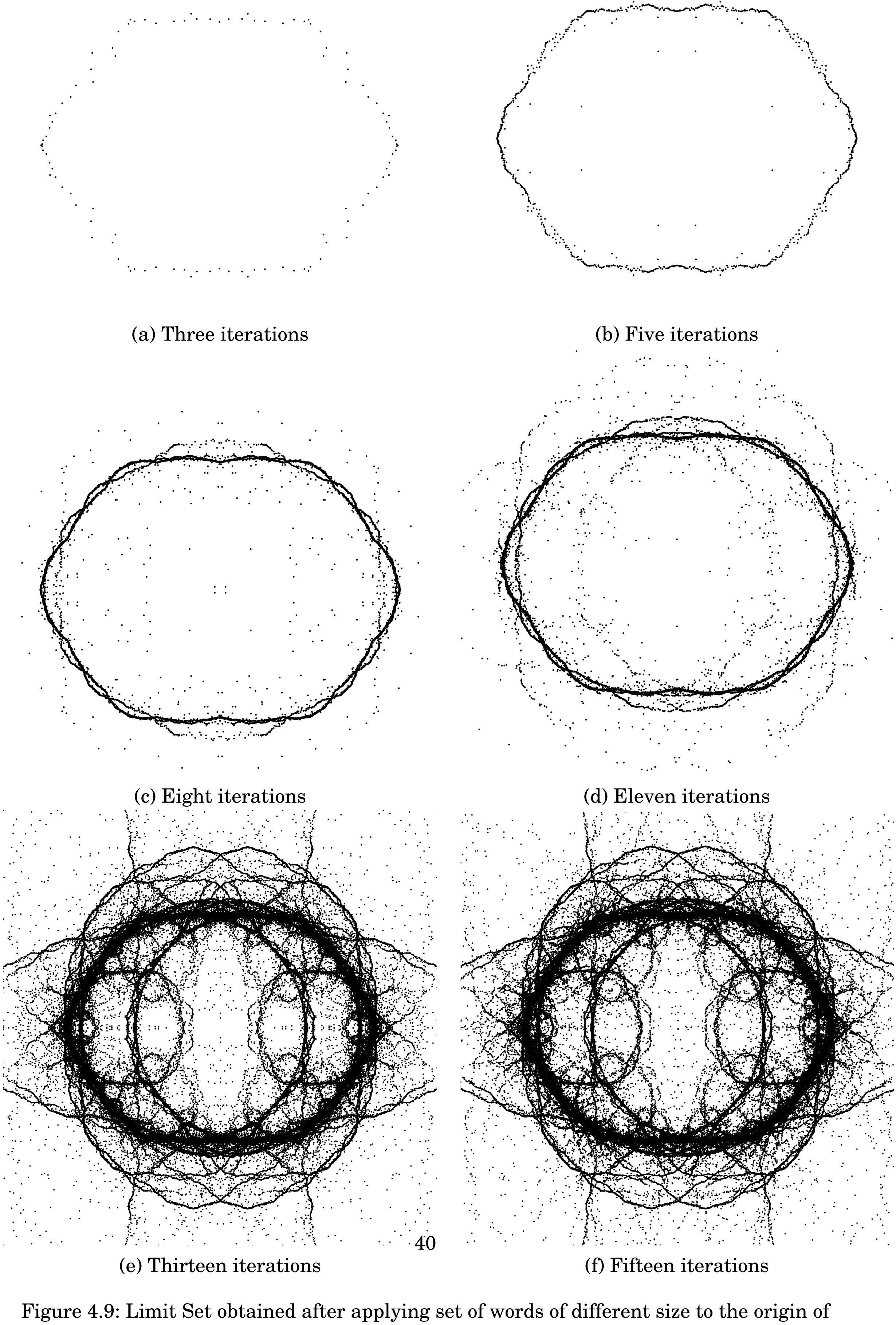
(a) After five iterations



(b) After five iterations

Figure 4.8: Limit set after a different number of iterations.

$z_4 = sz_3$, $w = z_5 + Q$. The slow convergence into the limit set can be visualized in Figure 4.9. With the numerical resources available to the author convergence was unfortunately not achieved.



(a) Three iterations

(b) Five iterations

(c) Eight iterations

(d) Eleven iterations

(e) Thirteen iterations

(f) Fifteen iterations

Figure 4.9: Limit Set obtained after applying set of words of different size to the origin of coordinates.

4.3 Some Topological Properties

We remind herein a couple of topology concepts: (i) The set $A \subset X$ is *dense* in X if every point x in X either belongs to A or is a limit point of A. Equivalently, being dense implies that the closure of A, $\bar{A}$, coincides with X. (ii) Similarly, a *nowhere dense set* is one whose closure has an empty interior.

Let us see how these concepts apply to the Limit Set.

Theorem 12. *Λ is nowhere dense in $\hat{\mathbb{C}}$.*

Proof. By the definition of the Limit Set, there are points of Ω^0 in every neighborhood of $x \in \Lambda$. But Ω^0 and Λ are separate, $\Omega^0 \cap \Lambda = \emptyset$, and thus the interior of Λ is empty. □

As a consequence of the previous theorem we have:

Theorem 13. *Either $\Lambda(G)$ is $\mathbb{S}^2$ or its interior is empty.*

Proof. This is a direct consequence of the proof of Theorem 12. There, it was substantiated that if there is a point $z \in \Omega^0$ then the interior of $\Lambda(G)$ is empty. If $\Omega^0 \neq \emptyset$ then $\Lambda(G) = \mathbb{S}^2$. □

Theorem 14. *Λ is closed.*

Proof. The proof relies upon the fact that a closed set contains all its limit points.

Let $\{x_m\}$ be a sequence of points in Λ, which converges to x. By the same arguments of the proof of Theorem 10, there is a single point $z \in \Omega^0$ and a sequence $\{g_{m,k}\}$ of distinct $g_{m,k}$ so that $g_{m,k}(z) \to x_m$.

Let δ_m be the minimal distance between x_m and any other $x_j \in \{x_m\}$. We now make the sub-index k of $g_{m,k}$ dependent on m, $k(m)$. The $k(m)$ are chosen so that the set of $\{g_{m,k(m)}\}$ is such that $d(g_{m,k(m)}(z), x_m) < \delta_m/2$. Therefore, $\{g_{m,k(m)}\}$ is a sequence of distinct elements of G, and because the $x_m \to x$ then $g_{m,k(m)}(z) \to x$. The last statement constitutes exactly the clause that we needed to prove. □

An isolated point of a set is one for which there exists a neighborhood which does not contain any other point of the set. A *perfect set* is a closed set with no isolated points. Every point can be approximated arbitrarily well by other points. That is, given any point and any neighborhood of the point, there is another point of the set within the neighborhood. This results in the fact that every point of the set is a point of accumulation of other points of the set.

Theorem 15. *If Λ contains more than two points, then it is perfect.*

Proof. Let us suppose that Λ contains at least three points. Let the district points be $x, y, z \in \Lambda$. For any limit point, x, there is a succession $\{g_m\}$ of distinct elements of G so that $g_m(z) \to x$. Note that $y \neq z$. Thus, by theorem 10, there are two district limit points, x_1 and x_2, not necessarily different from x, so that $g_m(x_1) \to x$ and $g_m(x_2) \to x$. For a particular m either $g_m(x_1) \neq x$ or $g_m(x_2) \neq x$. Therefore, there is sequence of limit points converging to any arbitrary point x. □

Definition 19 (Elementary Group). *A Kleinian group whose limit set consists of at most two points is called an Elementary group. Groups with Limits Sets containing more elements are called non-elementary.*

Theorem 16. *The G-orbit of any point in $\Lambda(G)$ is dense in $\Lambda(G)$.*

Proof. The proof is based on Montel's Theorem (Remmert, 2013), (Wikipedia, 2016). See Annex A for the theorem itself and some additional definitions.

Let us suppose that a point, ζ, is in $\Lambda(G)$ and let us denote with D an open disk centered at ζ. Supposing the sequence of distinct elements $\{A_n\} \in G$ has been normalized, for a $\omega \in \mathbb{S}^2$, we have $\zeta = \lim A_n(\omega)$.

We now take two $\zeta_1 \neq \zeta_2$ in $\Lambda(G)$. We state that the family $\{G\}$ of Möbius transformation acting in D cannot omit the two values ζ_1, ζ_2.

Let us assume otherwise, supposing therefore that the third statement of Montrel's Theorem, Theorem 33 on page 77 is fulfilled[2]. Under this hypothesis, $\{G\}$ must be normal. We look for a contradiction in the next few paragraphs to carry out the proof.

Let us reconsider the sequence $\{A_n\}$.For some index $N \in \mathbb{N}$ we have that $A_n(\omega) \in D$ for all $n \geq N$. Let us denote $\omega' = A_N(\omega)$ and set $B_n = A_n A_N^{-1}$. $\lim B_n(\omega') = \zeta$. We must also have convergence. Thus, $\lim B_n(z) = \zeta$, converging uniformly for z in compact subsets of D. Particularly, the image $B_n(D')$, of a sub-disk D' which verifies that $\zeta \in D' \subset \bar{D}' \subset D$, for large indexes n, is a proper subset of D'. Therefore, for all k, n, we have that $B_n^k(D') \subset B_n^{k-1}(D') \subset \ldots \subset D'$ which implies that B_n is loxodromic (for large n) with an attracting fixed point in D'. But for a fixed large n, the sequence $\{B_n^{-k}\}$ does not converge uniformly in compact sub-sets of D because it contains the repelling fixed points, a fact that implies a contradiction as sought.

We may now proceed to finish the proof. Let us consider now that for some $\xi \in \Lambda(G)$, ζ is not a limit point of the G-orbit $G(\xi)$. Then, there is a disk D centered at ζ that does not contain points of $G(\xi)$. The G-orbit does not meet either ξ or any other point on its orbit. Nevertheless, it has been proven immediately above that this was impossible. This means that $\Lambda(G)$ is dense[3]. □

Theorem 17. $\Lambda(G)$ *is the closure of the set of loxodromic fixed points, and if there are parabolic fixed points, it is the closure of the set of parabolic fixed points as well.*

Proof. G is non-elementary. Therefore, there are infinitely many distinct loxodromic transformations in G. If ξ is a fixed point of the loxodromic T, any point $A(\xi)$ on its G-Orbit is a fixed point of a loxodromic ATA^{-1}. The same happens if ξ is a parabolic fixed point.

[2]This third argument states that: A family of holomorphic functions, all of which omit the same two values $a, b \in \mathbb{C}$, is normal. A family F of holomorphic functions is called **normal** in a region D of $\mathbb{C}$ if every sequence of functions in F has a subsequence that converges compactly in D.

[3]In order to facilitate the reading: a subset $A \subset X$ is called dense in X if every point x in X either belongs to A or is a limit point of A.

Because Λ is closed and dense Theorems 14 and 16 Λ is equal to its closure. Let q_n be the attracting fixed point of the loxodromic T_m, see Theorem 16. $\zeta = \lim q_n$ is the limit of a sub sequence of positive powers $\{T_n^k(\omega)\}$ for any ω different from repulsive point $p = \lim p_n$. □

Example 4. *In our examples, we have seen that the limit set forms different kinds of patters depending on the generators utilized.*

In Figure 4.1 a Jordan curve has been shown. Figures 4.3 and 4.5 showed a limit set which consisted of isolated points. In this case, the isolation of the points was due to the fact that the circles utilized to obtain the generators were not tangent.

An even more curious result was obtained in Figure 4.9. For its generation, we tried to repeat the same procedure utilized for Figure 4.1, but we did not obtain a nice simple limit set in the form of a Jordan curve. We did find a scallop worthy of Hieronymus Bosch.

We will see now how parabolic fixed points and commutators combine together to deduce some conditions that must be fulfilled to obtain a Jordan curve as a limit set.

We will deduce them observing Figure 4.10. In the Figure it is depicted the usual pattern we utilize for circle pairing. Let us define that the circles C_1 and C_1' are paired by the transformation a. We also set that C_2 and C_2' are paired by transformation b. We also marked with P, Q, R and S the respective points of tangency of the circles.

Let us try to study the conditions that must be fulfilled to obtain a Limit Set of the type of the black line in the picture. To do this we follow Mumford et al. (2002).

1. *We utilize the rules of paring described in the Schottky group description, Definition 14 in Example 1 of page 28.*

2. *The circles must be tangent. Otherwise, the limit cannot be continuous.*

3. *To keep a continuous nice limit set the tangent points should be mapped following the rules*

$$a(R) = Q, \tag{4.1}$$

$$a(S) = P, \tag{4.2}$$

$$b(R) = S, \tag{4.3}$$

$$b(Q) = P. \tag{4.4}$$

These rules have a very interesting consequence.

$$aba^{-1}b^{-1}(P) = aba^{-1}(Q) = ab(R) = a(S) = P. \tag{4.5}$$

That is, $[a,b] = 1$. Clearly, all cyclic combinations of $aba^{-1}b^{-1}$ also commute. Thus, the other tangent points, Q, R, S, are also kept constant. We may check this utilizing $b^{-1}a^{-1}ba(R) = b^{-1}a^{-1}b(Q) = b^{-1}a^{-1}(P) = b^{-1}(S) = R$.

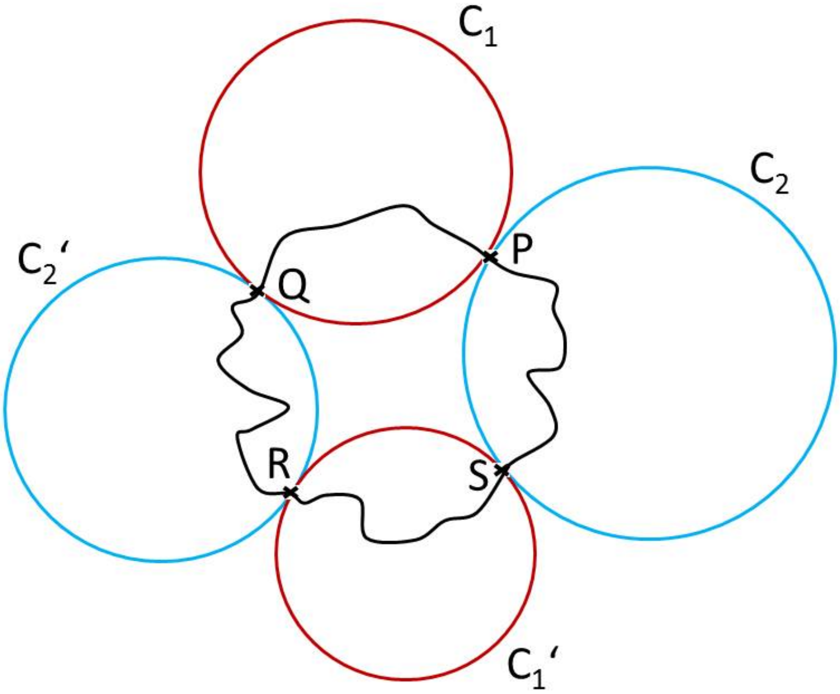


Figure 4.10: Sketch of the circle pairing for a two generator Schottky Group

4. *The Schottky circles in Figure 4.1 shrink down to a single point in all tangent points. This can be better observed in the detailed cut of Figure 4.11.*

   They actually nest down from both sides to a single point, being tangent to each other into the tangent point. Following Mumford et al. (2002), this means that the tangent points are parabolic. This can be characterized explicitly through

$$tr(aba^{-1}b^{-1}) = \pm 2. \tag{4.6}$$

   Now consider the normalized flavor of a *and* b. *We rewrite these generators as*

$$a(z) = kz, \tag{4.7}$$

   and

$$b(z) = \frac{az+b}{cz+d}. \tag{4.8}$$

   We calculate the commutator and find that

$$tr(aba^{-1}b^{-1}) = -\frac{bc(k-1)^2}{k} + 2 \tag{4.9}$$

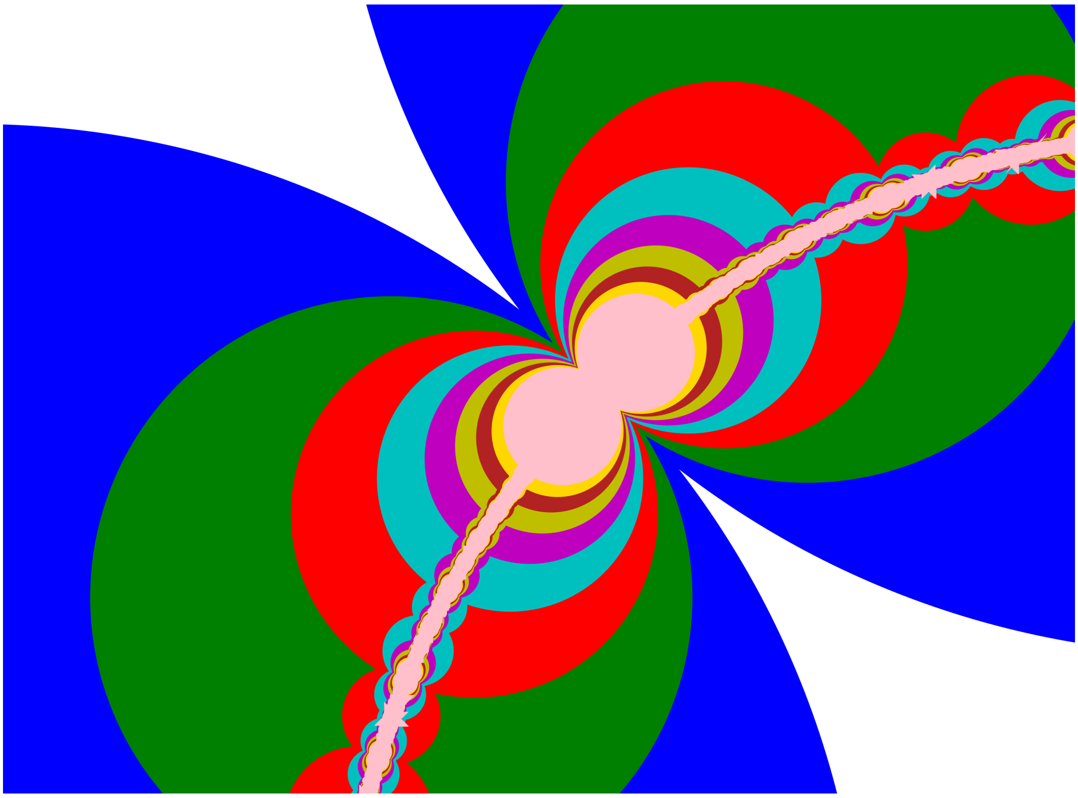


Figure 4.11: Circles nesting down.

We assume that $tr(aba^{-1}b^{-1}) = 2$. *We have*

$$\frac{bc(k-1)^2}{k} = 0, \tag{4.10}$$

which is only verified if $b = 0$, $c = 0$ *or* $k = 1$. *In either case the transformations have a common fixed point. For* $b = 0$ *the* 0, *for* $c = 0$ *the* ∞ *and for* $k = 1$ *the fixed points of* b. *This is nevertheless impossible with the pattern of circles depicted in Figure 4.10. Therefore,* $tr(aba^{-1}b^{-1}) = -2$.

Following Mumford et al. (2002), groups in which its verified that $tr(aba^{-1}b^{-1}) = -2$ *are called Parabolic Commutator Groups.*

Let continue now our analysis of the limit set.

Theorem 18. *If $D_1, D_2 \in \mathbb{S}^2$ are two open disks with disjoint closures and each of them meets $\Lambda(G)$, there exists a loxodromic element in G with a fixed point in D_1 and in D_2.*

Proof. We take two loxodromic transformations $A_1, A_2 \in G$. These transformations have their corresponding attracting points in the open disks D_1, D_2 respectively.

We now consider the repelling point of A_2. Two possibilities should be taken into account: whether its repelling point is located in D_1 or not.

1. If the repelling point is in D_1 the proof is finished.

2. In the second case, we take another loxodromic transformation, $h \in G$, with fixed points q_1, q_2 which are not equal to the corresponding ones of A_1, A_2.

   The rest of the proof is mainly based on the fact that the successive application of loxodromic transformations make points approach the attracting point of the loxodromic transformation. This fact is also applies to the inverse transformation. In this case the roles of the repelling and attracting point are simply inverted.

   The conjugate of h, $B_1 = A_1^m h A_1^{-m}$, with $m > 0$ has fixed points $A_1^m(q_1), A_1^m(q_2)$, see Figure 4.12. For sufficiently large m these fixed points both lie in D_1.

   We consider now a second disk D_1' located inside of D_1 which only contains the repelling point, p, but not the attracting point, q, of B_1. For a sufficiently large n, the transformation A_2^n sends q inside of D_2, see the diagram of Figure 4.12.

   At this stage, we need to prove that for a sufficiently large r, $A_2^n B_1^r$ verifies that,

$$A_2^n B_1^r(\overline{D_2}) \subset D_2, \tag{4.11}$$

$$B_1^{-r} A_2^{-n}(\overline{D_1'}) \subset D_1'. \tag{4.12}$$

   We undertake this without further delay.

   (i) For a large enough r, $B_1^r(D_2)$ is as closed to q as desired. Therefore, A_2^n sends the image $B_1^r(D_2)$ into D_2. Relationship 4.11 is proved.

   (ii) $A_2^n(q) \notin D_1'$. This means that $q \notin A_2^{-n}(D_1')$. This implies that r can be increased as much as necessary until it is verified that $B_1^{-r} A_2^{-n}(D_1')$ is included into D_1', as close as needed to the repelling point of B_1. This constitutes the proof of the relationship (4.12).

   The transformation $B_1^{-r} A_2^{-n}$ has two fixed points located as we needed, maps the disk as required and is clearly loxodromic. We have thus finished our proof.

□

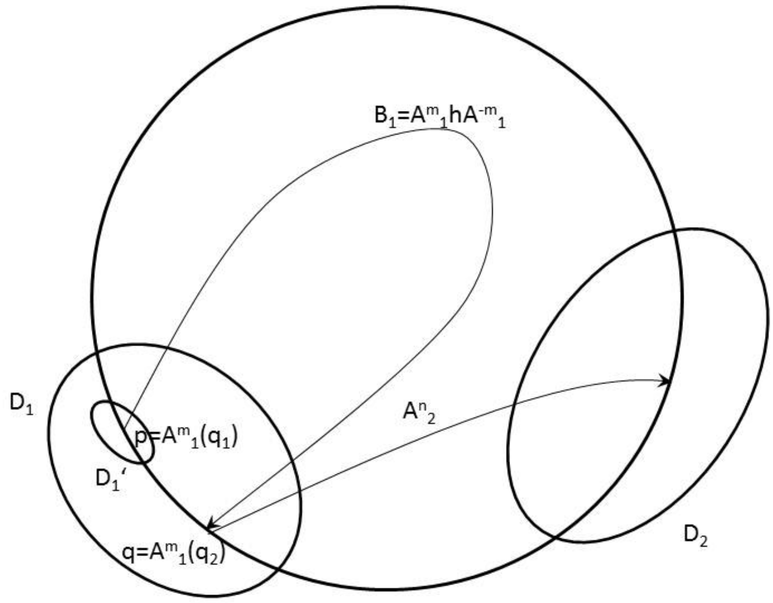


Figure 4.12: Sketch for the proof of Theorem 18

Theorem 19. *If G_0 has finite index in G then* $\Lambda(G_0) = \Lambda(G)$.

Proof. We consider a transformation $A \in G$ which is loxodromic. If G_0 has finite index in G, there is a $k \in \mathbb{N}$, $k > 0$, so that $A^k \in G_0$. Thus, $\Lambda(G_0)$ has the same set of loxodromic fixed points of $\Lambda(G)$. The closure of them is $\Lambda(G)$, see Theorem 17. As the closure of both is equal, then also $\Lambda(G_0) = \Lambda(G)$. □

Theorem 20. *If G_0 is a normal subgroup of G, then* $\Lambda(G_0) = \Lambda(G)$.

Proof. If G_0 is a normal subgroup of G, it is verified that $gG_0g^{-1} = G_0$, $\forall g \in G$. The image by g of the fixed points of a $h \in G_0$ are the fixed points of $ghg^{-1} \in G_0$, see also Section 2.1.4 and Theorem 17. Thus, $g\Lambda(G_0) = \Lambda(G_0)$ for all $g \in G$. But $\Lambda(G_0) \subset \Lambda(G)$. Considering that the G-orbit of the fixed points of a loxodromic transform $h \in G_0$ is, by theorem 16, dense in $\Lambda(G)$, this means that the Limit Sets are identical. □

4.4 The Complementary of the Limit Set

After the investigation of the Limit Set carried out in earlier sections of this chapter, it is somehow natural to extend our analysis to the complementary of the Limit Set.

In Definition 4 we have provided a strict concept of discreteness. We will consider a more gentle criterion here in order to study the complementary of the Limit Set.

Definition 20 (Discontinuity)**.** *A group G acts discontinuously at an $x \in X$ if there is a neighborhood U of x, so that $g(U) \cap U = \emptyset$ for all but finitely many $g \in G$.*

Definition 21 (Set of Discontinuity)**.** *The set of all points of X at which a group G acts discontinuously is called the set of discontinuity and is denoted by $\Omega(G)$.*

For completion, we provide now the reminder:

Definition 22. *The **stabilizer** of Y in G is the set,*

$$Stab_G(Y) = \{g \in G : g(Y) = Y\}. \tag{4.13}$$

Definition 23 (Precisely invariant)**.** *The set Y is precisely invariant under the subgroup H in G if,*

- $H = Stab_G(Y)$.
- $g(Y) \cap Y = \emptyset,\ \forall g \in (G - H)$.

Theorem 21. *Let G be Kleinian. A point x is in $\Omega(G)$, if and only if,*

1. *$Stab(x)$ if finite*
2. *There is a neighborhood U of x such that it is precisely invariant under $Stab(x)$.*

Proof. Let us prove firstly the sufficient condition. If $x \in \Omega(G)$, by Definition 20, $Stab_G(x)$ is finite. If $x \in \Omega(G)$, $g(U) \cap U \neq \emptyset$ for a finite number of transformations g. Let us take an $V \subset U$ so that $g(V) \cap V \neq \emptyset$ only for $g \in Stab_G(x)$. The intersection $\cap_{g \in Stab_G(x)} g(V)$ is a neighborhood which is precisely invariant under H.

We attempt the necessary condition. If $Stab_G(x)$ is finite and U is precisely invariant under $Stab(x)$, then $g(U) \cap U \neq \emptyset$ for the finite number of elements of $Stab(x)$. □

Theorem 22. *For any Kleinian group G, $\hat{\mathbb{C}}$ is the disjoint union of $\Lambda(G)$ and $\Omega(G)$.*

Proof. We will firstly prove that the intersection between $\Lambda(G)$ and $\Omega(G)$ is empty. Subsequently, we will prove that $\Lambda(G)$ and $\Omega(G)$ make a partition of $\hat{\mathbb{C}}$.

1. If $z \in \Lambda$, its orbit is dense (Theorem 16) in Λ. Thus, there are infinitely many distinct elements of $g \in G$ so that for any neighborhood, U, of z, $g(U) \cap U \neq \emptyset$. Therefore, $\Lambda(G) \cap \Omega(G) = \emptyset$. We have proven that the intersection is empty as required.

2. Let us consider a point $x \not\in \Omega(G)$. For every neighborhood U of x there are infinitely many translations of U, $g_m(U)$ with $m \in \mathbb{N}$ and $g_m \in G$, that intersect U. Between them we consider a sequence of elements $\{g_m\} \in G$ and a sequence of points $\{z_m\}$ such that $z_m \to x$ and $g_m(z_m) \to x$. By Theorem 10 there exists a subsequence of $\{g_m\}$ and there are two points w and y such that $g_m(z) \to w$ converges uniformly in compact sub-sets for z in $\hat{\mathbb{C}} - y$. w and y are both limit points. We have two possibilities to discuss: i) $x = y$; ii) $x \neq y$. If $x = y$ then, $x = y \in \Lambda(G)$. Otherwise, if $x \neq y$, then the points do not accumulate at y. We have proven the partition, because if $x \notin \Omega(G)$ we may take $x = w$ and it happens that them $x \in \Lambda(G)$.

□

Further interesting information relevant to the Set of Discontinuity can be found in Annex B. The contents outlined there sadly fit neither the narrative of this chapter nor the extent of this work. It contains some results for which extensive knowledge of Manifolds and their Covering, Riemann surfaces and some notions about the Fundamental Group are necessary.

Chapter 5

The Fractal Nature of the Limit Set

In previous sections we have characterized the limit set from the point of view of the Kleinian's group theory. In this chapter we shall mix our previous approach with some theory of fractals in order to describe the limit set from a complementary perspective. With this goal, we introduce some topics on measure and fractal dimension. After that, we turn our attention to the limit sets of Kleinian groups in order to study in which conditions can they be considered as fractals. We try to address this topic by investigating the local structure of the limit set and particularly the existence of tangents.

5.1 General

We may quote Mandelbrot (1982) in his famous essay The Fractal Geometry of Nature, to define that a **fractal** is a *a rough or fragmented geometric shape that can be split into parts, each of which is (at least approximately) a reduced-size copy of the whole*. This is the famous notion of *self similarity*, see also Vicsek (1992).

To complete our definition we may continue with another quotation from Mandelbrot (1982). *A fractal is by definition a set for which the Hausdorff-Besicovitch dimension strictly exceeds the topological dimension.*

This last definition puts an emphasis on dimension when treating fractals.

5.2 Measure and Dimension

5.2.1 General

We can address the concept of dimension based on an intermediate concept. Such is the concept of **measure**.

Definition 24 (Borel Set)**.** *A Borel set is a set, generated in terms of a topological space, such that can be created from open sets or, equivalently, from closed sets through countable union, intersection and relative complement.*

Definition 25. *μ is a measure in $\mathbb{R}^n$ if μ assigns a non-negative number, including ∞, to each subset of $\mathbb{R}^n$ and also verifies the following three properties:*

1. $\mu(\emptyset) = 0$.
2. *If $A \subset B$ then $\mu(A) \le \mu(B)$.*
3. *If A_1, A_2, ... is a countable sequence of sets then*

$$\mu\left(\cup_{i=1}^{\infty} A_i\right) \le \sum_{i=1}^{\infty} \mu(A_i) \tag{5.1}$$

A significant property of the measure that we will utilize in later sections is enunciated by the Egoroff's theorem.

Theorem 23 (Egoroff's theorem)**.** *We consider that:*

- *D is a Borel sub-set of $\mathbb{R}^n$.*
- *μ is a measure that accomplishes $\mu(D) < \infty$.*
- *$f_1, f_2, \ldots$ and f are functions from D to $\mathbb{R}$, verifying that for all $x \in D$, $f_k(x) \to f(x)$.*

In such conditions for every $\delta > 0$ there is a Borel sub-set $E \subset D$ such that $\mu(D \setminus E) < \delta$. Also, the sequence of $\{f_k\}$ converges uniformly to f on E.

The proof of this theorem can be found in detail in Sun (2016).

5.2.2 Topological dimension

Let us try to introduce the concept of dimension, providing a formal definition of the intuitive approach of considering a body d-dimensional if it *resembles* a patch of $\mathbb{R}^d$.

This we may reach following Engelking (1995), Arkhangel'Skii and Fedorchuk (2012) and Kohavi and Davdovich (2006), considering three definitions of the concept of dimension: the so called *Small inductive dimension*, the *Large inductive dimension* and the *Lebesgue covering* *dimension*. Among them, the latter is also known as *topological dimension* and it is the concept we will introduce.

We remind that a **cover** of a set X is a collection of sets whose union contains X as a subset. Also, that a refinement of a cover is a new cover such that every set in the refinement is contained in some set of the original cover.

Definition 26. ***Lebesgue covering dimension*** *also called generically,* ***topological dimension****, is the minimum value n, such that any open cover has a refinement in which no point is included in more than n+1 elements of the refinement.*

If no minimal exists then the dimension is infinite. A set is zero-dimensional if every open cover of the space has a refinement consisting of disjoint open sets. All points are contained in only one element of the refinement.

A convenient refinement for $\mathbb{R}^2$ is given in Figure 5.1 illustrating the concept and showing that each point is included in at most three elements of the refinement.

We do not extend further in this topic of Topological dimension and refer for further details to e.g. the monograph work of Engelking (1995).

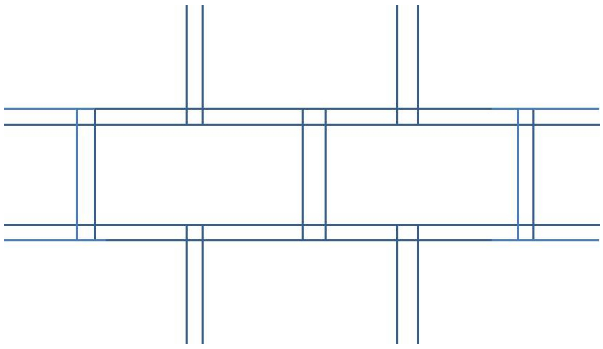


Figure 5.1: Lebesgue covering refinement

5.2.3 Hausdorff dimension

We address now the definition of the α-dimensional Hausdorff measure. To do this we firstly define the set,

$$\Lambda_\alpha^\epsilon(X) = \left(\inf \sum_k r_k^\alpha : r_k < \epsilon; X \subset \cup D_k(x_k, r_k)\right), \tag{5.2}$$

where α is a non-negative number, the infimum is taken over all covers $\{D_k\}$ of X and $D_k(x_k, r_k)$ is a disk of center x_k and radius r_k of at most a size ϵ.

Definition 27 (α-dimensional Hausdorff measure)**.** *The α-dimensional Hausdorff measure, Λ_α, of a set $X \subset \mathbb{C}$, is defined in terms of the set $\Lambda_\alpha^\epsilon(X)$ as*

$$\Lambda_\alpha(X) = \lim_{\epsilon \to 0} \Lambda_\alpha^\epsilon(X). \tag{5.3}$$

If the diameter of each member of a cover is smaller than a certain value, that we denote by δ, we call this cover a δ-*cover*.

The literature consulted provides small variations in the definition of α-dimensional Hausdorff measure given here. Concretely, different sources consider X a closed or plain set or a sub-set of $\mathbb{R}^n$. Also, $\{D_k\}$ can be seen as a plain or an open cover.

The Hausdorff Dimension can be defined in terms on the α-dimensional Hausdorff measure.

Definition 28 (Hausdorff or Hausdorff-Besicovitch Dimension)**.** *The Hausdorff Dimension is defined as,*

$$\dim X = \inf\{\alpha : \Lambda_\alpha(X) = 0\}.$$

The role of the infimum in the previous definition is not evident. One should realize that for smaller values of α, $\Lambda_\alpha(X) = \infty$. Actually,

$$\Lambda_\alpha(X) = \begin{cases} \infty & if\ \alpha < dimX \\ 0 & if\ \alpha > dimX \end{cases} \tag{5.4}$$

As such, we can define the Hausdorff-Besicovitch Dimension alternatively as $\dim X = \sup\{\alpha : \Lambda_\alpha(X) = \infty\}$.

We now consider the inverse path and suppose we know that the Hausdorff-Besicovitch dimension of a certain set, X, is α. We can thus define a related concept:

Definition 29. *Let X be a Borel set and $dimX = \alpha$. X is an* ***α-set*** *if it is verified that* $0 < \Lambda_\alpha(X) < \infty$.

The definition above excludes the cases in which $dimX = \alpha$ but either $\Lambda_\alpha(X) = 0$ or $\Lambda_\alpha(X) = \infty$.

Very often an alternative definition of Hausdorff dimension is given:

Definition 30 (Box-dimension)**.** *Given an $\epsilon > 0$ we take $N(\epsilon)$ as the number of balls of diameter ϵ that completely cover a set X. The* ***box dimension*** *of X is,*

$$dim_B(X) = \lim_{\epsilon \to 0} \frac{\log N(\epsilon)}{\log(1/\epsilon)}. \tag{5.5}$$

The logarithm simply defines the power law at which the amount of balls grows with the descent of the diameter. This power law represents actually the value we were looking for.

Other alternative definitions of dimensions exist, see e.g. Falconer (2004, chapter 3).

Example 5. ***The Cantor Set.*** *Let's look now at a well known example to illustrate previous concepts.*

The ***Cantor set*** *consists of the closed points in the unit interval whose triadic expansion does not contain any occurrences of the the digit 1, i.e.,*

$$X = \left\{ \sum_{k=1}^{\infty} \frac{i_k}{3^k} : i_k = \{0,2\}, k \geq 1 \right\}. \tag{5.6}$$

The Cantor set is represented in Figure 5.2.

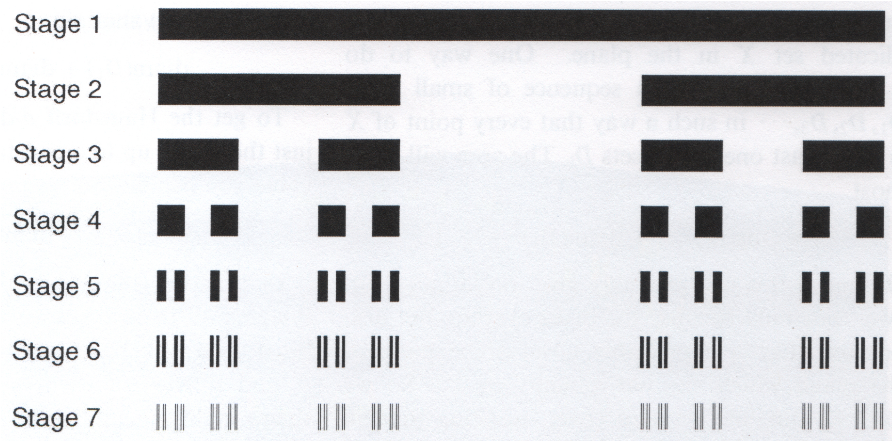


Figure 5.2: Several stages of the Cantor set. Picture obtained from Mandelbrot (1982)

We now look for an adequate cover in order to calculate the box dimension. We may consider a certain value $\epsilon_n = 1/3^n$. *We may cover the set* X *by the union of* $2n$ *intervals,*

$$X_n = \left\{ \sum_{k=1}^{n} \frac{i_k}{3^n} + \frac{t}{3^n} : i_k = \{0,2\}, k \geq 1, 0 \leq t \leq 1 \right\} \tag{5.7}$$

of extension $1/3^n$. *This implies that* $N(\epsilon_n) \leq 2^n$.

Note that any interval of length 2^n *intersecting the set* X *can touch at most two intervals from* X_n. *This means that* $N(\epsilon_n) \geq 2^{n-1}$.

Let us now take another value ϵ, *of a magnitude such that it verifies that* $\epsilon_{n+1} \leq \epsilon \leq \epsilon_n$. *Therefore,* $N(\epsilon_n) \leq N(\epsilon) \leq N(\epsilon_{n+1})$. *Thus,*

$$\frac{n-1}{n+1}\frac{\log 2}{\log 3} \leq \frac{\log(N(\epsilon_n))}{\log(1/\epsilon_{n+1})} \leq \frac{\log(N(\epsilon))}{\log(1/\epsilon)} \leq \frac{\log(N(\epsilon_{n+1}))}{\log(1/\epsilon_n)} \leq \frac{n+1}{n}\frac{\log 2}{\log 3}. \tag{5.8}$$

Making $n \to \infty$, *the box dimension results in* $\log 2/\log 3$.

We may also note that the analogy between the Cantor Set of Figure 5.2 and the limit set obtained in Figure 4.3 and 4.5 is significant.

5.3 Hausdorff Dimension Properties

We provide in this section several properties of the Hausdorff dimension that we shall utilize later. We start by restating two definitions.

Definition 31. $L : X_1 \to X_2$ *is a Lipschitz map if there exists* $C > 0$ *such that* $|L(x)-L(y)| \leq C|x-y|$.

Definition 32. $L : X_1 \to X_2$ *is a Bi-Lipschitz map if there exists* $C > 0$ *such that* $(1/C)|x-y| \leq |L(x)-L(y)| \leq C|x-y|$.

Based on these definitions, we may see that:

Theorem 24. *If $L : X_1 \to X_2$ is a surjective Lipschitz map, then $dim(X_1) \leq dim(X_2)$.*

Theorem 25. *If $L : X_1 \to X_2$ is a bijective Bi-Lipschitz map, then $dim(X_1) = dim(X_2)$.*

Proofs of Theorems 24 and 25. For the Theorem 24, consider an open cover U, one that has all its subsets open, for X_1 with $diam(U_i) \leq \epsilon$ for all subsets $U_i \in U$. Then the images $U' = L(U_i) : U_i \in U$ are a cover for X_2 with $diam(L(U_i)) \leq C\epsilon$ for all $U_i \in U'$ and a $C > 0$. Thus, from the definitions, $\Lambda^{\epsilon}_{\alpha}(X_2) \geq \Lambda^{\epsilon}_{\alpha}(X_1)$. In particular, letting $\epsilon \to 0$ we see that $\Lambda_{\alpha}(X_2) \geq \Lambda_{\alpha}(X_1)$. Finally, from the definitions $dim(X_1) \leq dim(X_2)$. For the proof of theorem 25, we can apply the first part a second time with L replaced by L^{-1}. □

We now study the dimension of an ensemble of sets.

Theorem 26. *Let $\Lambda_1, \Lambda_2 \subset \mathbb{R}$ and let $\Lambda_1 + \Lambda_2 = \{\lambda_1 + \lambda_2 : \lambda_1 \in \Lambda_1, \lambda_2 \in \Lambda_2\}$ then $dim(\Lambda_1 + \Lambda_2) \leq dim(\Lambda_1) + dim(\Lambda_2)$.*

Proof. It is immediate that $dim(\Lambda_1 \times \Lambda_2) = dim(\Lambda_1) + dim(\Lambda_2)$. We may combine this with the map $L(x, y) = x + y$ which is Lipschitz to obtain the result. □

Based in the definitions above we can now derive an important clause: the so called Hölder condition.

Proposition 4. *Let $X \subset \mathbb{R}^n$ and $f : X \to \mathbb{R}^m$. Let f verify,*

$$|f(x) - f(y)| \leq c\,|x - y|^{\beta} \tag{5.9}$$

for the points $x, y \in X$ and let c and β be constants such that $c > 0$ and $\beta > 0$. For each α we have

$$\Lambda_{\alpha/\beta}(f(X)) \leq c^{\alpha/\beta}\Lambda_{\alpha}(X). \tag{5.10}$$

Proof. Let $\{U_i\}$ be a δ-cover of X. $\text{diam}(|f(X \cap U_i)) \leq c\text{diam}(|U_i)^{\beta}$. Take $\epsilon = c\delta^{\beta}$. $\{f(X \cap U_i)\}$ is a ϵ-cover of $F(X)$. Therefore, $\sum_i \text{diam}(f(X \cap U_i))^{\alpha/\beta} \leq c^{\alpha/\beta}\sum_i \text{diam}(U_i)^{\alpha}$, so that $\Lambda^{\epsilon}_{\alpha/\beta}(f(X)) \leq c^{\alpha/\beta}\Lambda^{\delta}_{\alpha}(X)$. Taking $\delta \to 0$, also $\epsilon \to 0$ which gives the desired result. □

Of a particular importance is the result in which f is a Lipschitz mapping and $\beta = 1$. We have,

$$\Lambda_{\alpha}(f(X)) \leq c^{\alpha}\Lambda_{\alpha}(X). \tag{5.11}$$

5.4 The Fractal Nature of the Limit Set

Our interest focuses now on the fascinating picture of Felix Klein, Figure 1.5. In this illustration, a limit set with a singular topology is shown.

It would be interesting to analyze whether this singular topology was serendipity or if it is a fundamental characteristic of the limit set. In this section we strive to uncover this, at least partially.

Example 6. *We may indeed give a further indication that the limit set is a fractal taking into account the definitions of previous sections. This we can do with a couple of interesting examples.*

Maybe one of the better-known fractals that can be constructed utilizing the methodology described in the examples 1, 2, 3 and 4 is the so called Apollonian Gasket. The Apollonian Gasket is a very notable figure of great beauty that has been studied among others by Bourke (2006), Mandelbrot (1982) and Mumford et al. (2002) and that is also relevant for problems of circle packing, see Graham et al. (2005).

For the construction of this limit set we utilize the set of circles shown in Figure 5.3. This corresponds to a circle C_1 of infinite radius, two of radius one, C_2 and C_2', tangent to each other and to the fourth circle C_1. C_1' is located in between the others, and is tangent to each of them as appears in the Figure 5.3.

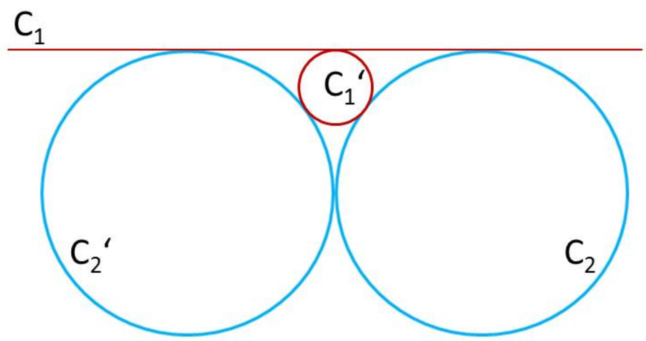


Figure 5.3: Schema of the circles utilized to obtain of the generators of the Apollonial Gasket.

The generators of the Apollonian Gasket are well known, see Mumford et al. (2002). The application of the methodology of example 4 for the calculation of the generators yields the values of

$$a = \begin{pmatrix} 1 & 0 \\ -2i & 1 \end{pmatrix} \tag{5.12}$$

and

$$b = \begin{pmatrix} 1-i & 1 \\ 1 & 1+i \end{pmatrix}. \tag{5.13}$$

The code of example 3 with these new generators immediately provides Figure 5.4. Note the distinctive convergence of the points in different locations of its limit set. Notably, close to the points of tangency of the generators (parabolic, near the horizontal axis of the figure) and their images, the limit set appears quite void.

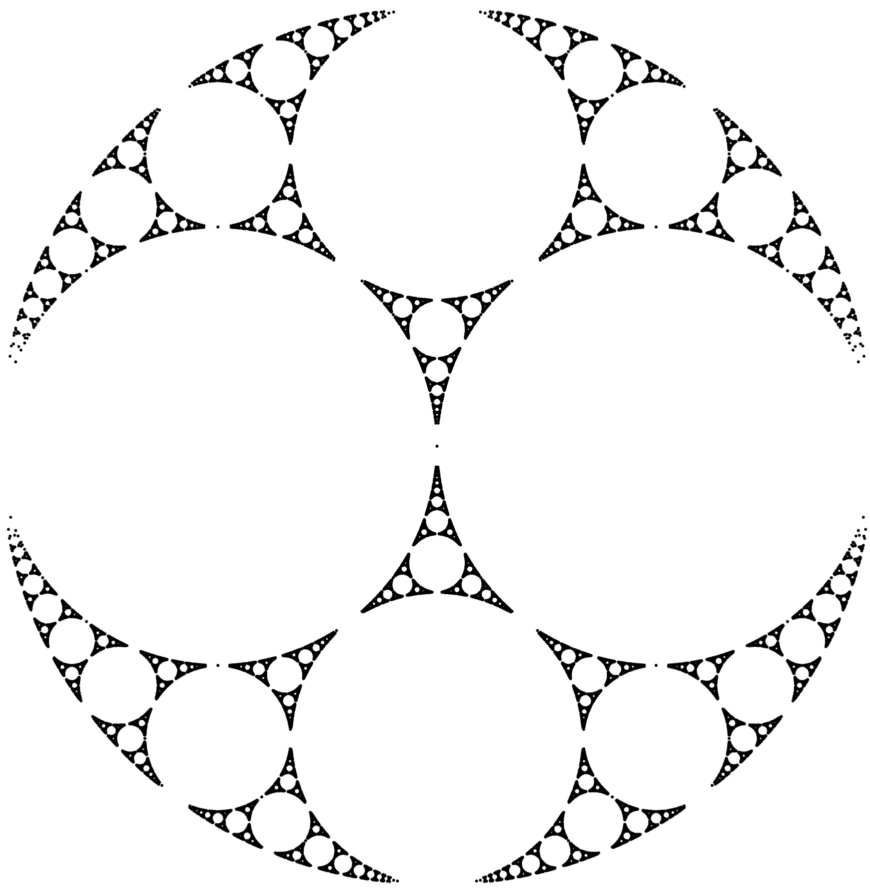


Figure 5.4: The Apollonial Gasket.

An interesting variation of the last figure happens when the point of tangency between the circles C_2 and C_2' is located at ∞. The arrangement of the circles becomes as it appears in Figure 5.5. In this case, the limit set adopts an interesting configuration in form of a stripe, see Figure 5.6. To obtain the figure, we have utilized the same code used to calculate Figure 5.4 but with different generators, obtained again from Mumford et al. (2002),

$$a = \begin{pmatrix} 2 & -i \\ -i & 0 \end{pmatrix}, \tag{5.14}$$

and

$$b = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}. \tag{5.15}$$

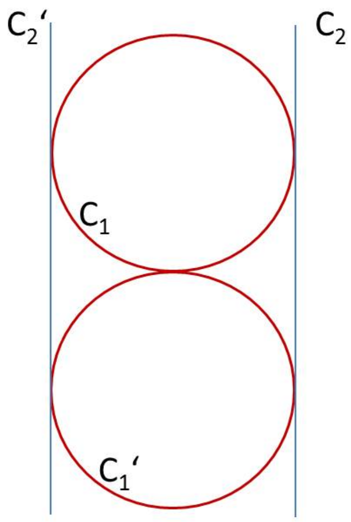


Figure 5.5: Schema of the circles utilized for obtaining the generators of the Apollonial Gasket in stripe form.

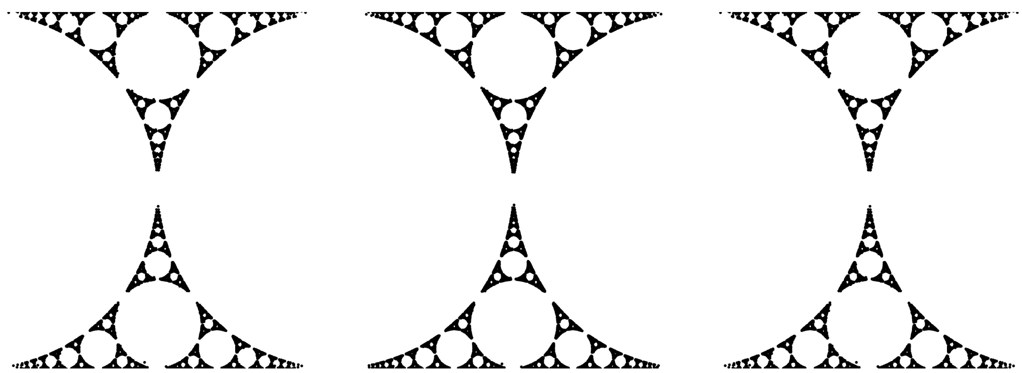


Figure 5.6: The Apollonial Gasket in form of a stripe.

At this moment it would be appropriate to summarize some points about the kind of sets we are dealing with and about the pertinence of calling them fractals. To do so–and for the sake of a general overview–we will advance some results that will be adequately proven later on.

Thus: (i) We are investigating limit sets that have been obtained by recursion. (ii) The limit set of elementary groups consists of two points by definition. For all others cases not involving elementary groups: (iii) We will see in Theorem 32 on page 67 that there are no smooth limit sets except circles. Euclidean lines, conceived as circles with infinite radii, are a particular case of smooth limit sets. In these cases, we would be dealing with Fuchsian groups. (iv) We will also prove in the same theorem that each component of $\Lambda(G)$ which is neither a point, nor a circle, nor a line has nowhere a tangent.

We are considering complex sets with intricate topologies. Regarding the pertinence of calling them fractals, we may recall the definition given by Marden (2007):

Definition 33. *A connected closed set without an interior which has a Hausdorff dimension larger than one is called a **fractal**.*

This of course constitutes an *ad hoc* definition for fractals appearing in the specific kind of problems we are treating.

For elementary groups, we may find the following theorem to be interesting.

Theorem 27. *For any countable set X, $dim(X) = 0$.*

Proof. We start enumerating the countable set $X = \{x_n : n \geq 1\}$. Given any $\alpha > 0$ and $\epsilon > 0$, for each $n \geq 1$, we can choose $\epsilon > \epsilon_n > 0$ sufficiently small that $\sum \epsilon_n^{\delta} = \epsilon$. We can consider the cover U for X by balls $B(x_n, \epsilon_n/2)$ centered at x_n and of different diameters ϵ_n. From the definitions, $\Lambda_\alpha^\epsilon(X) \leq \epsilon$, for any ϵ and so $\Lambda_\alpha(X) = 0$. As $\alpha > 0$ was arbitrarily chosen, from the definition of Hausdorff dimension, it is verified that $dim(X) = 0$. □

And thus the Hausdorff dimension for the elementary group is zero. The dimension of a line is one. Thus the Hausdorff dimension of the limit set of Fuchsian group is one. For the rest of the cases, we may advance that the Hausdorff-Besikovich dimension coincides with the so called *Poincare or Critical exponent* and refer for more details to the specific literature on the topic, Nicholls (1989), Bishop and Jones (1997) and Canary et al. (1994). We can treat it therefore as a α-set with this dimension.

5.5 The Local Structure of the Limit Set

Making an analogy with the structure of *usual* curves, we may want to study the approximation of our limit set through tangent lines. Nevertheless, is some cases the limit set exhibits a complex structure and the mere existence of a tangent is not evident. We have already seen some of those intricate structures in previous examples.

Therefore, one may try to qualify the structure of the limit set through two characteristics: (i) The first could be to study its local density. That is, how the limit set is concentrated around some locations. This is of particular significance for the cases in which the limit set is formed by a set of points, for example in Figure 4.5. (ii) The second is the directional distribution of the limit set. This specially refers to the existence or non-existence of a tangent.

5.6 Densities

Definition 34. *The **density** of a subset, X, of the plane in a point $x \in X$ is*

$$\lim_{r\to 0} \frac{area(X \cap B_r(x))}{area(B_r(x))}, \tag{5.16}$$

where $B_r(x)$ is the closed disk of radius r and center x.

Definition 35. *A **coordinate parallelepiped** in $\mathbb{R}^n$ is the set*

$$A = \{(x_i, \ldots, x_n) \in \mathbb{R}^n : a_i \leq x_i \leq b_i\}, \tag{5.17}$$

where $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ are two points of $\mathbb{R}^n$.

Definition 36. *The n-dimensional volume of the coordinate parallelepiped A is given by*

$$vol^n(A) = (b_1 - a_1) \cdot \ldots \cdot (b_n - a_n) \tag{5.18}$$

Definition 37. *The n-dimensional Lebesgue measure of the subset X of $\mathbb{R}^n$ is given by*

$$L^n(X) = inf \left\{ \sum_{i=1}^{\infty} vol^n(A_i) : X \subset \cup_{i=1}^{\infty} A_i \right\}. \tag{5.19}$$

The infimum is taken over all covering of X by coordinate parallelepipeds A_i.

Note that $L^n(A) = vol^n(A)$ if A is a *usual* set, that is, a coordinate parallelepiped.

The Lebesgue Density theorem, see Mattila (1999), establishes that the density of any Lebesgue measurable Borel set X–a Borel set where a Lebesgue measure has been stablished– is 1 when $x \in X$ and 0 when $x \notin X$, except for a set around x of area 0. This has a simple interpretation. For a point $x \in X$, small balls arround x are almost filled by X and vice versa for those cases in which $x \notin X$. Similarly,

$$\lim_{r\to 0} \frac{Length(X \cap B_r(x))}{2r} = 1 \tag{5.20}$$

if $x \in X$ and 0 if $x \notin X$.

We are now going to extend the concept of density to an α-set. We consider firstly a function $f : \mathbb{R}^+ \to \mathbb{R}$. We intend to characterize this function for small positive numbers. At the same time, we suspect that f may fluctuate wildly close to zero and that the limit will not necessarily exist. To characterize such variations we provide two definitions:

Definition 38. *The lower limit of $f(x)$ is*

$$\underline{\lim}_{x\to 0} f(x) = \lim_{r\to 0}(\inf\{f(x) : 0 < x < r\}). \tag{5.21}$$

Definition 39. *The upper limit is given by*

$$\overline{\lim}_{x\to 0} f(x) = \lim_{r\to 0}(\sup\{f(x) : 0 < x < r\}). \tag{5.22}$$

If both upper and lower limits coincide the *conventional* limit exists and equals its value.

We are now in a position to extend the concept of density to α-sets through the concept of upper and lower densities.

Definition 40. *The lower and upper densities of a α-set X at a point $x \in \mathbb{R}^n$ are given by the equations*

$$\underline{D}^{\alpha}(X,x) = \underline{\lim}_{r\to 0}\frac{\Lambda_\alpha(X \cap B_r(x))}{(2r)^\alpha}, \tag{5.23}$$

$$\overline{D}^{\alpha}(X,x) = \overline{\lim}_{r\to 0}\frac{\Lambda_\alpha(X \cap B_r(x))}{(2r)^\alpha}. \tag{5.24}$$

Lower and upper densities allow for a classification of the points and α-sets. If $\underline{D}^{\alpha}(X,x) = \overline{D}^{\alpha}(X,x) = 1$ the point is called *regular*, or *irregular* otherwise. We denote the common value by $\underline{D}^{\alpha}(X,x) = \overline{D}^{\alpha}(X,x) = D^{\alpha}(X,x)$.

We say that an α-set is Λ^α**-almost** if all of its points, except for a set of measure 0, verify a property.

We classify now sets analogously to points. We say that a α-set X is called *regular* if is Λ^α-almost regular or *irregular* otherwise.

The classification above not only provides a taxonomy of points and α-sets. It has also implications, that we will study later, which concern the existence of a tangent.

We provide a couple of properties of the densities.

Theorem 28. *X is a α-set. Then, it is verified that:*

1. $\underline{D}^{\alpha}(X,x) = \overline{D}^{\alpha}(X,x) = 0$ *for Λ^α-almost all $x \notin X$.*
2. $2^{-\alpha} \le \overline{D}^{\alpha}(X,x) \le 1$ *for Λ^α-almost all $x \in X$.*

The proof of these properties exceeds the scope of this document, and consequently readers are referred to Falconer (2004).

Definition 41. *A cluster point, y in a subset E, is one for which there exist other points of the subset as close as desired (arbitrarily).*

For α-sets in which $0 < \alpha < 1$ the following theorem is worth noting.

Theorem 29. *If X is an α-set in $\mathbb{R}^2$ then X is irregular unless α is an integer.*

Partial proof. We suppose that the density $D(X,x)$ exist everywhere in X and look for a contradiction to this fact.

We continue our rationale based on this assumption. There is a set $X_1 \subset X$ of positive measure where the density exits and where, following Theorem 28, it is verified that $1/2 < 2^{-\alpha} \leq D^\alpha(X,x)$. By the Egoroff Theorem (Th. 23 in page 52) exists an $r_0 > 0$ and a Borel set $E \subset X_1 \subset X$ with $\Lambda_\alpha > 0$ such that

$$\Lambda_\alpha(X \cap B_r(x)) > (2r)^\alpha/2 \tag{5.25}$$

for all $x \in E$ and $r < r_0$.

We now take $y \in E$, a cluster point. Let η be a number with $0 < \eta < 1$. We consider the annulus $A_{r,\eta} = B_{r(1+\eta)}(y) \setminus B_{r(1-\eta)}(y)$. Then we have

$$\frac{\Lambda_\alpha(X \cap A_{r,\eta})}{(2r)^\alpha} = \frac{\Lambda_\alpha(X \cap B_{r(1+\eta)})}{(2r)^\alpha} - \frac{\Lambda_\alpha(X \cap B_{r(1-\eta)})}{(2r)^\alpha}. \tag{5.26}$$

By the definition of density, Definition 40, the RHS of previous equation converges to

$$D^\alpha(X,y)\big((1+\eta)^\alpha - (1-\eta)^\alpha\big) \tag{5.27}$$

as $r \to 0$.

Now we consider a sequence of values of r which tends to 0. We may find a point x in E such that it is verified that $|x-y| = r$. Therefore, $B_{r\eta/2} \subset A_{r,\eta}$. Taking into account equation (5.25), we obtain

$$\frac{1}{2} r^\alpha \eta^\alpha < \Lambda_\alpha(X \cap B_{r\eta/2}) \leq \Lambda_\alpha(X \cap A_{r,\eta}). \tag{5.28}$$

Considering equation (5.27) we get

$$\begin{aligned} 2^{-\alpha-1}\eta^\alpha \leq & D^\alpha(X,y)\big((1+\eta)^\alpha - (1-\eta)^\alpha\big) \\ = & D^\alpha(X,y)\big(2\alpha\eta + o(\eta^2)\big). \end{aligned} \tag{5.29}$$

Making $\eta \to 0$, last statement becomes impossible when $\alpha < 1$. We have obtained the contradiction we were looking for. □

5.7 Tangents to the Limit Set

We turn our attention now to a particular local property of the limit set. We may try to study the nature of the tangents of the limit sets of Kleinian Groups. For this section we utilize specific literature on the limit sets of Kleinian Groups, like Marden (2007) or Lehto (1987), complementing the sources on fractals.

We start analyzing an interesting particular case: the limit set of Fuchsian group.

5.7.1 The Limit Set of Fuchsian Groups

We provide an apparently *ad hoc* definition of Fuchsian Group.

Definition 42. *A Kleinian group is Fuchsian if it keeps invariant some circular disk, D.*

The pertinence of the example in Figure 4.1 is now underlined by the statement above.

There are certainly different approaches that can be followed for the definition of Fuchsian groups.

We emulate e.g. Beardon (1983), Lehto (1987), Marden (2007) and Maskit (1988) defining the Fuchsian groups as subgroups of the Kleinians.

A distinct approach can be prosecuted by defining Fuchsian groups directly as a discrete subgroup of $PSL(2\mathbb{R})$. This last possibility is followed by other authors like e.g. Jones and Singerman (1987). In this latter reference contains an introductory but complete review of the properties of Fuchsian groups.

The next property is a surprising characteristic of the limit set that will allow us to classify Fuchsian groups.

Theorem 30. *For any non-elementary group G, the limit set Λ is the smallest non-empty G-Invariant subset of $\hat{\mathbb{C}}$.*

Proof. Let E be any non-empty, closed G-invariant subset of $\hat{\mathbb{C}}$. If G is non-elementary, every orbit is infinite, so that E is infinite. Let v be any point fixed by a loxodromic element $g \in G$. Therefore, there is some w in E, not fixed by g and the set $\{g^n(w)\}$ with $n \in \mathbb{N}$, that accumulate at v. $v \in E$ because E is closed. Thus, the set of points fixed by loxodromic elements in G is contained in E. The limit set is the closure of the former set. As E is closed is also included in E. Thus, $\Lambda \subseteq E$, for all possible E. We have concluded our proof. □

The direct application of Theorem 30 to the definition of Fuchsian groups implies that their limit set is contained in the circle ∂D. This is the result that we wanted to prove and that motivates the formulation of Theorem 30. By Definition 42 we have described a group by stating the shape of its limit set.

We may actually classify the Fuchsian groups based on this result.

Definition 43. *Let G be a Fuchsian group with an invariant disk D. G is said to be of the **first kind** if $\Lambda = \partial D$ strictly and of the **second kind** if $\Lambda \subset \partial D$ properly.*

Related to Fuchsian groups, we may define a akin concept, in which the limit set is not a circle but a Jordan Curve.

Definition 44. *A Quasi-Fuchsian group is a Kleinian group whose limit set is a Jordan Curve.*

Please recall that Jordan Curves were already defined in page 30.

Example 7. *We may now continue the rationale of Example 4. Our objective is to obtain a Jordan curve as a limit set.*

With this goal in mind, we will prove a property of the trace of the commutator that we will utilize immediately in this example.

Theorem 31. *The trace of the commutator is,*

$$Tr(aba^{-1}b^{-1}) = (Tr(a))^2 + (Tr(b))^2 + (Tr(ab))^2 - Tr(a)Tr(b)Tr(ab) - 2 \tag{5.30}$$

where a and b are the Matrix equivalent of two Möbius transformations,

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \tag{5.31}$$

and

$$\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}. \tag{5.32}$$

Proof. We prove this equality following the methodology outlined by Fricke and Klein (1897). We obtain the products of the elements of equations (5.31) and (5.32) in equation (5.30). For this tedious task, we utilize the mathematical symbolic computation program *Mathematica* by Wolfram Research, Inc. (2010). We obtain,

$$\begin{aligned} &Tr(aba^{-1}b^{-1}) - [(Tr(a))^2 + (Tr(b))^2 + (Tr(ab))^2 - Tr(a)Tr(b)Tr(ab) - 2] = \\ &2 - a_1^2 - a_2^2 - a_2^2 b_1 c_1 - a_1^2 b_2 c_2 - 2b_1 b_2 c_1 c_2 - 2a_1 d_1 + \\ &a_1 a_2^2 d_1 - d_1^2 - b_2 c_2 d_1^2 - 2a_2 d_2 + a_1^2 a_2 d_2 + 2a_1 a_2 d_1 d_2 + a_2 d_1^2 d_2 \\ &-d_2^2 - b_1 c_1 d_2^2 + a_1 d_1 d_2^2. \end{aligned} \tag{5.33}$$

We may regroup the terms on the right hand side to obtain,

$$\begin{aligned} RHS = &a_1^2(-1 - b_2 c_2 + a_2 b_2) + a_2^2(-1 - b_1 c_1 + a_1 d_1) \\ &+ d_1^2(-1 - b_2 c_2 + a_2 d_2) + d_2^2(-1 - b_1 c_1 + a_1 d_1) \\ &2 - 2b_1 b_2 c_1 c_2 - 2a_1 d_1 - 2a_2 d_2 + 2a_1 a_2 d_1 d_2 \end{aligned} \tag{5.34}$$

The terms of the last equation inside of the brackets cancel out, as our matrices a and b have a determinant equal to one. For the rest of the terms, we apply again that $a_1 d_1 - 1 = c_1 b_1$. We expand $b_1 b_2 c_1 c_2 = a_1 d_1 a_2 d_2 - a_1 d_1 - a_2 d_2 + 1$. Substituting the last equation in equation (5.34) we see immediately that it cancels out. □

We are now prepared to study the generic problem of the generation of a limit set in the form of a Jordan curve based on generators in the form of equations (5.31) *and* (5.32).

Because $det(a) = det(b) = 1$, this is a six parametric problem. We may still reduce the number of parameters by means of normalization. We consider that a, a^{-1} and b have attracting fixed points in $0, 1, \infty$ respectively. This reduces the number of parameters to three.

We now bring in an additional consideration. We have seen in example 4 that $tr(aba^{-1}b^{-1}) = -2$. Therefore, equation (5.30) simplifies to

$$(Tr(a))^2 + (Tr(b))^2 + (Tr(ab))^2 = Tr(a)Tr(b)Tr(ab). \tag{5.35}$$

This corresponds to an additional constraint that reduces the number of parameters to two. In fact, considering the format of the last equation, it appears conceivable to utilize $Tr(a)$ and $Tr(b)$ as the two parameters of the system, and resolve generators a and b as a function of $Tr(a)$ and $Tr(b)$.

This was the strategy followed by Mumford et al. (2002). Utilizing equation (5.35) as a second order equation in variable $Tr(ab)$, this last magnitude is obtained. Once $Tr(ab)$ is known, the generators adopt a very elegant formulation. For simplicity we denote $Tr(a) = t_a$, $Tr(b) = t_b$ and $Tr(ab) = t_{ab}$. The generators become, following Mumford et al. (2002),

$$a = \begin{pmatrix} t_a/2 & \dfrac{(t_a t_{ab} - 2t_b + 4i)(t_b t_{ab} - 2t_a + 2it_{ab})}{(2t_{ab} + 4)(t_{ab} - 2)t_b} \\ \dfrac{(t_a t_{ab} - 2t_b + 4i)(t_{ab} - 2)t_b}{(2t_{ab} + 4)(t_b t_{ab} - 2t_a + 2it_{ab})} & t_a/2 \end{pmatrix}, \tag{5.36}$$

and

$$b = \begin{pmatrix} \dfrac{t_b - 2i}{2} & t_b/2 \\ t_a/2 & \dfrac{t_b + 2i}{2} \end{pmatrix}. \tag{5.37}$$

We are now in a position to calculate a limit set that will be a Jordan Curve different from the circle of Figure 4.1.

For the completion of this task we take the program already utilized in example 4. We carry out a run with a set of generators calculated with equations (5.36) and (5.37). For this example, the values $t_a = t_b = 2.7$ are considered. The Figure 5.7 is obtained.

5.7.2 Tangents to the Limit Set of a Generic Kleinian group

We consider here the classical definition of the tangent as the line through a pair of infinitely close points on a curve or set. More concretely, if we consider a curve $y = f(x)$ and a point $x = c$ on the curve we are interested in the line passing through the point $(c, f(c))$ which has a slope $f'(c)$ where f' is the derivative of f. As usual, we are thus defining the tangent in terms of the derivative, that is the limit $\lim_{h\to 0}(f(c+h) - f(c))/h$. It is well known that in many cases such limit does not exist.

We will obtain now a very important result for our analysis and the characterization of the limit set.

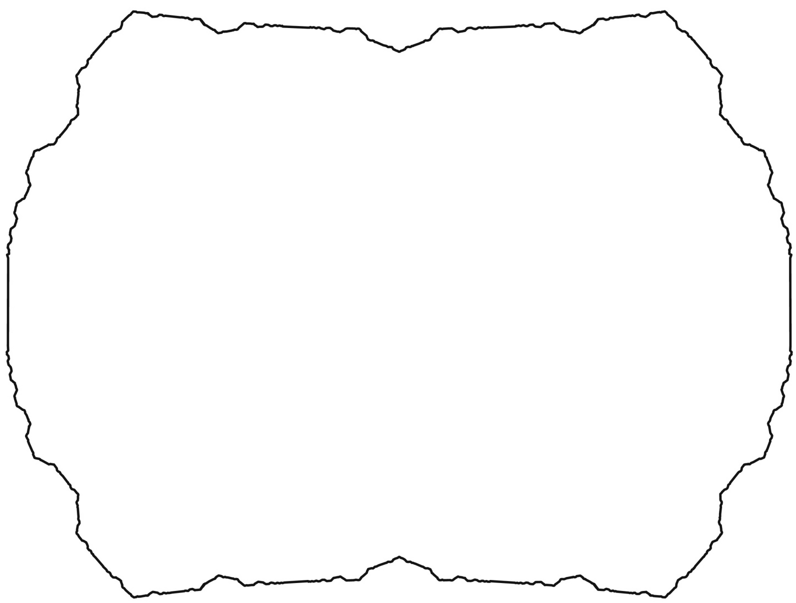


Figure 5.7: Limit set in form of Jordan curve.

Theorem 32. *A limit set $\Lambda(G)$ cannot have a tangent line at a a fixed point of a loxodromic transformation $g \in G$ unless $\Lambda(G)$ is Möbius equivalent to a circle.*

This theorem is attributed to Lehto (1987), although the formulation we are following comes from Marden (2007).

Lehto was investigating Quasi-Fuchsian groups–like the one in Example 7–and the formulation he utilizes was more adapted to the goals he was pursuing. We give the original formulation (Lehto, 1987) for completion:

Let G be a Kleinian group such that its Set of Discontinuity has an invariant component A which is a Jordan domain–the one circumvented by a Jordan curve–*different from a disk. Then, the border of the Jordan domain, ∂A, does not have a tangent at a fixed point of a loxodromic element of G.*

Proof of Theorem 32. Let us assume that the tangent exists at a fixed point of a loxodromic element $g \in G$. We may look for a contradiction of this hypothesis.

We suppose, without any kind of loss of generality, that the fixed point z, lies in $z = 0$. To achieve this condition, we may normalize if needed. Also because of normalization, it can be supposed that the tangent at z is the real axis, and that the repulsive fixed point lies at ∞.

We write down g in its generic form, $g(z) = ke^{i\varphi}z$, so that $0 < k < 1$ and $0 \leq \varphi < 2\pi$.

We divide the proof into two cases, $\varphi \neq 0, \pi$ and $\varphi = 0, \pi$.

(i) Let us start with the former case, $\varphi \neq 0$ or $\varphi \neq \pi$. The angle ϕ is set as,

$$\phi = \min\{\varphi, |\pi - \varphi|, 2\pi - \varphi\}. \tag{5.38}$$

We may construct the symmetric wedges of angle ϕ centered in z and along $\mathbb{R}$, see Figure 5.8. Those wedges are $V = \{re^{i\theta} : \theta \in (-\phi/2, \phi/2)\}$ and $V' = \{re^{i\theta} : \theta \in (\pi - \phi/2, \pi + \phi/2)\}$. Certainly, $0 < \phi \leq \pi/2$.

Let us now take a disk D, centered in z and with a sufficiently small radio. We have,

$$\Lambda \cap D \subset (V \cup V') \cap D.$$

Now, let us take a point $x \in \Lambda \cap D$. Because of theorem 11, and considering z as the attractive fixed point of g, it happens that $g(x) \in \Lambda \cap D$. On the other side, due to the definition of g, the argument of $g(x)$ is too large and it is not in $(V \cup V')$. We have arrived to a contradiction which proves the theorem for this case (i).

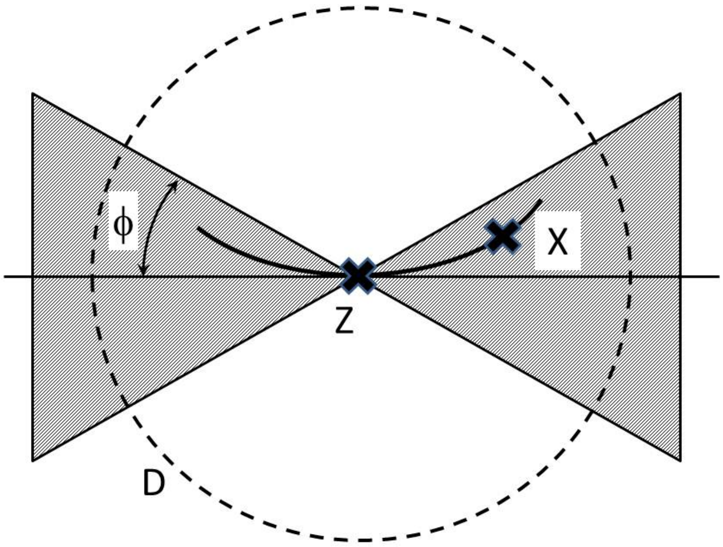


Figure 5.8: Schema of the domains and position of the points selected for the proof of the theorem.

(ii) Let us now turn our attention to the latter case, $\varphi = 0$. We have $g(x) = kx$. Take a point x in Λ so that $\Im x \neq 0$. Thus $\forall n \in \mathbb{N}$, $\arg g^n(x) = \arg x$. We have obtained another contradiction. It comes from the fact that $g^n(x) \in \Lambda$ is as close as desired from z. But then again, a line joining $g^n(x)$ and z never approximates the real axis. If $\varphi = \pi$, the argument is the same, but operating in terms of g^2.

□

5.7.3 Tangents to α-sets

The concept of tangent provides information on how a curve, or a set, is concentrated into two diametrically opposed directions.

We have already considered in previous sections the classical definition of the tangent as the line through a pair of infinitely close points on a curve or set. We found out that with the sets of the complexity we are treating, there is no hope of providing positive results able to characterize the orientation of the limit set involving all points in a certain vicinity.

This directional characterization can nevertheless be reached considering α-sets. With this aim, we will generalize the concept of tangent.

Definition 45 (Tangent of α-sets)**.** *The α-set $X \in \mathbb{R}$ has a tangent at x in direction $\vec{\theta}$ if*

$$\overline{D}^{\alpha}(X,x) > 0, \tag{5.39}$$

and for every angle $\phi > 0$,

$$\lim_{r\to 0} \frac{\Lambda_\alpha\left(X \cap \left(B_r(x) \setminus S(x,\vec{\theta},\phi)\right)\right)}{r^\alpha} = 0 \tag{5.40}$$

where $S(x,\vec{\theta},\phi)$ is the double sector with vertex x consisting of points y such that the segment $[x,y]$ makes an angle at most ϕ with $\vec{\theta}$ or $-\vec{\theta}$.

$S(x,\vec{\theta},\phi)$ constitutes a generalization of the wedges V and V' of the proof of Theorem 32, see also Figure 5.8.

This definition of tangent implies that the most of the α-set is oriented in $\pm\vec{\theta}$ direction while a negligible part is allowed to lie in other directions even in the near vicinity of point x.

5.7.4 Tangents to 1-sets

We consider the length $\mathscr{L}$ of a curve C given by a polygonal approximation $\mathscr{L}(C) = \sup \sum_{i=1}^{m} |x_i - x_{i-1}|$. The supremum is taken on all divisions of C by points $x_0, \dots, x_m$ in that order.

Definition 46. *C is called rectifiable if $\mathscr{L}(C)$ is finite.*

$\mathscr{L}(C)$ equals its 1-dimensional Hausdorff measure:

Lemma 2. *If C is a rectifiable curve then $\Lambda_1(C) = \mathscr{L}(C)$*

Proof. We consider points $x, y \in C$. By $C_{x,y}$ we denote the part of C between x and y. Orthogonal projection onto lines passing through x and y do not increase distances. We also consider the result of equation (5.11). Therefore,

$$\Lambda_1(C_{x,y}) \geq \Lambda_1([x,y]) = |x-y|, \tag{5.41}$$

with $[x,y]$ the straight line joining x to y. For any division in points $x_0, x_1, \ldots, , x_m$ of C,

$$\sum_{i=1}^{m} |x_i - x_{i-1}| \leq \sum_{i=1}^{m} \Lambda_1 \left(C_{x_i, x_{i-1}}\right) \leq \Lambda_1(C). \tag{5.42}$$

Thus, $\mathscr{L}(C) \leq \Lambda_1(C)$.

Now, we define a mapping $f : [0, \mathscr{L}(C)] \to C$ that takes a certain t in the interval $[0, \mathscr{L}(C)]$ to the point on C at a distance t along the curve from one of its ends. We have $|f(t) - f(u)| \leq |t-u|$ for $0 \leq t$, $u \leq \mathscr{L}(C)$ and we have, by equation (5.11), that $\Lambda_1(C) \leq \mathscr{L}(C)$. □

Lemma 3. *A rectifiable curve is a regular 1-set.*

Proof. C is rectifiable. This means that $\mathscr{L}(C) < \infty$. Let us consider that C has endpoints p and q. We have $\mathscr{L}(C) \geq |p-q|$. By Lemma 2, we have $0 < \Lambda_1(C) < \infty$. Thus C is a 1-set.

We consider now a point $x \in C_{p,q}$ not coinciding with p or q. It divides C into two parts, $C_{p,x}$ and $C_{x,q}$. We take another point $y \in C_{x,q}$ and a number r such that $|x-y| = r$. For r sufficiently small it is verified that $C_{x,y} \subset B_r(x)$ and

$$r = |x-y| \leq \mathscr{L}(C_{x,y}) = \Lambda_1(C_{x,y}) \leq \Lambda_1(C_{x,q} \cap B_r(x)). \tag{5.43}$$

By the same rationale,

$$r \leq \Lambda_1(C_{p,x} \cap B_r(x)). \tag{5.44}$$

Adding both equations, for small enough r we obtain

$$2r \leq \Lambda_1(C \cap B_r(x)). \tag{5.45}$$

This yields,

$$\underline{D}^1(C,x) = \underline{\lim}_{r\to 0} \frac{\Lambda_1(C \cup B_r(x))}{2r} \geq 1. \tag{5.46}$$

On the other hand, by Theorem 28,

$$\underline{D}^1(C,x) \leq \overline{D}^1(C,x) \leq 1. \tag{5.47}$$

$D^1(C,x)$ exists and equals 1 for all $x \in C$ other than the endpoints. C is thus regular. □

Proposition 5. *A rectifiable curve C has a tangent at almost all of its points.*

Proof. We start this proof establishing that due to Lemma 3, for almost all $x \in C$, $\overline{D}^1(C,x) = 1$.

We parametrize C by arc-length through a function φ. Therefore, $\varphi : [0, \mathscr{L}(C)] \to \mathbb{R}^2$ allows to calculate the coordinates of a point of C, $\varphi(t)$, as a function of the distance from the endpoint $\varphi(0)$. $\mathscr{L}(C) < \infty$ means that φ has a bounded variation. But functions of bounded variation are differentiable almost everywhere. This result can be found e.g. in Hewitt and Stromberg (1975, Chapter 17). $\varphi'(t)$ exists as a vector for almost all t. Even more, because of the parametrization$\left|\varphi'(t)\right| = 1$ for such t. At almost all points of $\varphi(t)$ on C there exists a vector $\vec{\theta}$ that verifies that $\lim_{u\to t}\left(\varphi(u)-\varphi(t)\right)/(u-t) = \vec{\theta}$.

Considering an arbitrary angle $\phi > 0$, there is a number $\epsilon > 0$ such that, whenever $|u-t| < \epsilon$, $\varphi(u) \in S(\varphi(t), \vec{\theta}, \phi)$. C has no double points . Thus, we may find a number r such that $\varphi(u) \notin B_r(\varphi(t))$ if $|u-t| \geq \epsilon$, so that $C \cap \left(B_r(\varphi(t)) \setminus S(\varphi(t), \vec{\theta}, \phi)\right)$ is empty. By the definition of tangent to an α-set, Definition 45, the curve C has a tangent at $B_r(\varphi(t))$. Those points are almost all the points on C. □

We prove now that regular 1-sets are analogous to *classical* curves and support tangents.

Proposition 6. *A regular 1-set X in $\mathbb{R}^2$ has a tangent at almost all of its points.*

Proof. If X is regular, $\overline{D}^1(X,x) = 1$ at almost all $x \in X$. We keep in mind our new definition of tangent, Definition 45, and specially formula (5.40), establishing that $\lim_{r\to 0}\left(\Lambda_\alpha\left(X \cap \left(B_r(x) \setminus S(x, \vec{\theta}, \phi)\right)\right)\right)/r^\alpha = 0$

Let us take C as a rectifiable curve. As given by Proposition 5, for almost all $x \in C$ there exists $\vec{\theta}$ such that for $\varphi > 0$,

$$\lim_{r\to 0}\frac{\Lambda_1\left((X\cap C)\cap\left(B_r(x)\setminus S(x,\vec{\theta},\phi)\right)\right)}{r} \leq \lim_{r\to 0}\frac{\Lambda_1\left(C\cap\left(B_r(x)\setminus S(x,\vec{\theta},\phi)\right)\right)}{r} = 0. \tag{5.48}$$

By Theorem 28, for almost all $x \in C$,

$$\lim_{r\to 0}\frac{\Lambda_1\left((X\setminus C)\cap\left(B_r(x)\setminus S(x,\vec{\theta},\phi)\right)\right)}{r} \leq \lim_{r\to 0}\frac{\Lambda_1\left((X\setminus C)\cap B_r(x)\right)}{r} = 0 \tag{5.49}$$

Adding inequalities (5.48) and (5.49), for almost all $x \in C$ and for almost all $x \in X \cap C$.

$$\lim_{r\to 0}\frac{\Lambda_1\left(X\cap\left(B_r(x)\setminus S(x,\vec{\theta},\phi)\right)\right)}{r} = 0. \tag{5.50}$$

A countable collection of C covers almost all X. Thus, we have finished our proof. □

On the other side, irregular 1-sets do not generally allow for tangents.

Proposition 7. *At almost all points of an irregular 1-set no tangets exist.*

The proof applying to this Proposition is too complex to be covered here. It can be followed in Besicovitch (1938, theorem 9, page 331 and discusions before)[1].

[1]Available online the Springer site of *Mathematische Annalen*.

5.7.5 Tangents α-sets

We consider now α-sets on the plane with α non-integer. First of all, we need to realize that by Theorem 29 we are considering exclusively irregular sets. On the plane, we consider two cases: (i) $0 < \alpha < 1$ (ii) $1 < \alpha < 2$.

For the first case, Falconer (2004) reports examples of α-sets supporting or not supporting tangents. Particularly, he underlines that a set with $0 < \alpha < 1$ contained in a smooth curve will automatically satisfy equation (5.40). It supports thus a tangent in most of its points.

For the second case an interesting–and general–result is available.

Proposition 8. *If X is an α-set in $\mathbb{R}^2$ with $1 < \alpha < 2$, then at almost all points of F, no tangent exists.*

Proof. Taken from Falconer (2004), which refers to and takes the proof from Marstrand (1954)[2]. As in previous results, e.g. the proof of Proposition 6, we need to keep in mind the form of our definition of tangent. The notation in this proof is slightly more complex than in the previous results although the same in nature.

We take a number $r_0 > 0$ and create a set,

$$E = \left\{ y \in X : \Lambda_\alpha(X \cap B_r(y)) < 2(2r)^\alpha \ \forall r < r_0 \right\}. \tag{5.51}$$

We intend now to make an estimation of the amount of E that lies in $B_r(x) \cap S(x, \vec{\theta}, \phi)$ with $x \in X$ and $0 < \phi < \pi/2$.

We consider a $r < r_0/20$, the intersection A_i of an annulus $B_{ir\phi} \setminus B_{(i-1)r\phi}$ and the double sector $S(x, \vec{\theta}, \phi)$,

$$A_i = \left(B_{ir\phi} \setminus B_{(i-1)r\phi}\right) \cap S(x, \vec{\theta}, \phi), \tag{5.52}$$

with $i = 1, 2, \ldots$. Thus, there is an integer $m < 2/\phi$ such that

$$B_r(x) \cap S(x, \vec{\theta}, \phi) \subset \cup_{i=1}^m A_i \cup \{x\}. \tag{5.53}$$

A_i has two separate parts with diameter of at most $10r\phi < r_0$. Applying equation (5.51) to the parts that contain points of E and summing,

$$\Lambda_\alpha \left(E \cap B_r(x) \cap S(x, \vec{\theta}, \phi) \right) \leq (4\phi^{-1}) 2(20r\phi)^\alpha. \tag{5.54}$$

If $r < r_0/20$ we obtain,

$$(2r)^{-\alpha} \Lambda_\alpha \left(E \cap B_r(x) \cap S(x, \vec{\theta}, \phi) \right) \leq 8 \cdot 10^\alpha \phi^{\alpha-1} \tag{5.55}$$

[2] Reference available online in spite of its publication date.

For almost all $x \in E$, it is verified that $\overline{D}^{\alpha}(X \setminus E, x) = 0$. We now split $X \cap B_r(x)$ into three parts, to obtain

$$\begin{aligned} \Lambda_\alpha(X \cap B_r(x)) = \Lambda_\alpha((X \setminus E) \cap B_r(x)) + \Lambda_\alpha\left(E \cap B_r(x) \cap S(x, \vec{\theta}, \phi)\right) \\ + \Lambda_\alpha\left(E \cap \left(B_r(x) \setminus S(x, \vec{\theta}, \phi)\right)\right). \end{aligned} \tag{5.56}$$

Dividing by $(2r)^\alpha$, operating, taking upper limits as $r \to 0$ for almost all $x \in E$

$$\overline{D}^{\alpha}(X, x) \leq 0 + 8 \cdot 10^\alpha \phi\alpha - 1 + \overline{\lim}_{r\to 0}(2r)^{-\alpha}\Lambda_\alpha\left(X \cap \left(B_r(x) \setminus S(x, \vec{\theta}, \phi)\right)\right). \tag{5.57}$$

If we consider a small enough ϕ, neither (5.39) nor (5.40) hold for any $\vec{\theta}$. Therefore, no tangent exists at x. We only need now to generalize the result to the whole set. By Theorem 28, for a $r_0 > 0$ almost all $x \in X$ belongs to E. We have finished our proof.

□

Therefore, we can state that our nice Apollonial gasket has not tangents even with the extended concept of tangents of Definition 45.

Chapter 6

Conclusions

We start underlining a surprising fact. Although a priori it can be considered as a niche thematic, the limit sets of Kleinian groups can be studied from a broad spectrum of approaches. An extensive overview of the matter requires, between others, a study from the point of view of the Riemann Surfaces, Hyperbolic Geometry, Covering Spaces and Group Theory. Although those matters are an integrating part of the corpus of this Master, the inclusion of all approaches inside of the frames of a single work is impossible. In the opinion of the author, this extends to the specialized literature on the topic. Not a single work among the available sources reviewed covers all these possible approaches exhaustively.

Therefore, we have concentrated in few topics. Firstly we have carried out the study of the limit set from a topological point of view. In this sense we may summarize the results that were found out:
(i) The intersection of the Free Regular Set and the Limit Set, Λ is empty. (ii) Let x be a limit point of a Kleinian Group G. There is a (second) limit point y, not necessary distinct to x, and a sequence of $\{g_m\}$ of distinct elements of G so that $g_m(z) \to x$ converges uniformly on compact subsets of $\hat{\mathbb{C}} \setminus \{y\}$. (iii) Λ is G-Invariant. (iv) Λ is nowhere dense in $\hat{\mathbb{C}}$. (v) Either $\Lambda(G)$ is $\mathbb{S}^2$ or its interior is empty. (vi) Λ is closed. (vii) If Λ contains more than two points then it is perfect. (viii) The G-orbit of any point in $\Lambda(G)$ is dense in $\Lambda(G)$. (ix) $\Lambda(G)$ is the closure of the set of loxodromic fixed points, and if there are parabolic fixed points, $\Lambda(G)$ is the closure of the set of parabolic fixed points as well. (x) If $D_1, D_2 \in \mathbb{S}^2$ are two open disks with disjoint closures, each of which meets $\Lambda(G)$, there exists a loxodromic element in G with a fixed point in D_1 and in D_2. (xi) If G_0 has finite index in G, then $\Lambda(G_0) = \Lambda(G)$. (xii) If G_0 is normal subgroup of G, then $\Lambda(G_0) = \Lambda(G)$. (xiii) Λ is the smallest non-empty G-Invariant subset of $\hat{\mathbb{C}}$.

Secondly, we have studied the local properties of the limit set of the Kleinian groups. We have striven to determine under what circumstances the limit set can be considered a fractal. It was found out that Kleinian groups that are neither elementary nor Fuchsian have a limit set that is not *smooth*. By *smooth* we understand that the limit set does not have a tangent at any of its limit points. Fuchsian groups have a limit let that is Möbius equivalent to a circle to a circle. Elementary groups have two points as limits. Equivalently, we may reformulate this using the words of Marden (2007) stating that *each component of the limit set which is not a circle or a point is a fractal set*.

The fractal limit sets can be characterized directionally utilizing an extended definition of tangent adequate for these sets. Some significant results on this alternative approach are contained in Chapter 5. As a major conclusion, it can be summarized that for sets with a Hausdorff-Besicovitch dimension larger than one and smaller than two for almost all points of the limit set no tangents exists–even considering the extended definition of tangent–. Those sets do not have a preferred directional orientation.

The properties summarized above have allowed us to implement an efficient and simple code with which we were able to calculate and generate limit sets. This was utilized to gain an increased comprehension on the formation of limit set: it has allowed us to clarify the results we were investigating by means of convenient examples. The examples cover the formation of limit sets illustrating the diversity of topologies that limit sets of Kleinian groups can reach.

Probably, the most interesting future work that can be carried out in order to improve the understanding of the limit sets of Kleinian groups is the study and characterization of their most defining property: the Hausdorff-Besicovitch dimension. The Hausdorff-Besikovich dimension coincides with the *Critical exponent* see Marden (2007), Nicholls (1989), Bishop and Jones (1997) and Canary et al. (1994). This analysis requires the extension of this work to the hyperbolic space where an enhanced comprehension of the convergence can be simply gained. Pitifully, a full comprehensive study of this notable property and the apparatus necessary for its deployment exceeds the frames of this document.

APPENDIX A

Montel's Theorem

In this Annex, we make a small digression concerning Montel's Theorem which was a necessary part of the proof of Theorem 16 in page 42.

In order to come closer to the topic, we need to introduce some concepts in the form of definitions. In this sense, we remind the reader that the definitions of Locally Compact Space and Compact Convergence have been already given on page 26.

Definition 47 (Bounded family). *A family, F, of holomorphic functions in the domain $D \subset \mathbb{C}$ is bounded in a subset $A \in D$ if there exists a real number, $M > 0$, such that $|f|_A < M$ for all $f \in F$.*

Definition 48 (Locally bounded). *A family, F, of holomorphic functions in a region $D \subset \mathbb{C}$ is locally bounded in D if for every point z of D there is a neighborhood U inside of D such that F is bounded in U.*

Definition 49 (Normal family). *(Remmert, 2013, page 152) A family F of holomorphic functions is called **normal** in a region D of $\mathbb{C}$ if every sequence of functions in F has a subsequence that converges compactly in D.*

After these definitions, we are in a position to be able to enunciate Montel's theorem.

Theorem 33. *Montel's Theorem (Remmert, 2013) (Wikipedia, 2016) states:*

- *A locally bounded family of holomorphic functions defined in an open subset of the complex numbers is normal.*
- *F is a family of meromorphic functions in an open set D. If $z_0 \in D$ is such that F is not normal at z_0, and $U \subset D$ is a neighborhood of z_0, then $\bigcup_{f \in F} f(U)$ is dense on the complex plane.*

- *A family of holomorphic functions, in which all members omit the same two values $a, b \in \mathbb{C}$, is normal.*

The proof of this theorem which exceeds the frames of this work can be found in Remmert (2013), or in Markushevich and Silverman (2005).

APPENDIX B

THE SET OF DISCONTINUITY REVISITED

We introduce in this annex some interesting results regarding the set of discontinuity that do not properly fit in the dedicated Section 4.4.

An enhanced analysis of the $\Omega(G)$ set requires one to delve into new subjects. Concretely, we will need an intermediary result. This is the so called *Ahlfors Finiteness theorem*.

This theorem and its proof require several concepts that have not been treated in this document. This is due to the necessary extension required for a serious treatment of the topics which is not possible considering the context in which this document has been written.

Those mainly are: Manifolds and their Covering; Riemann Surfaces; and some notions about the Fundamental Group. These issues have been treated among the different subjects in this Master of Advanced Mathematics. In any case, we allow ourselves to mention that a very interesting insight into these thematics can be gained by consulting Perez-Alvarez (2013), Bujalance-Garcia et al. (2003) and Hatcher (2002). We also mention that we will utilize concepts like quotient spaces and punctures, which we consider to be part of the previous subjects. An exception will be the concept of *conical point* for which we refer readers to Nicholls (1989).

Let us address without further ado the Ahlfors Finiteness theorem.

Theorem 34. *If G is a finitely generated Kleinian group, $\delta M(G) = \Omega(G)/G$ is the union of a finite number of (Riemann) surfaces. Each of them is a closed surface with at most a finite number of punctures and elliptic conical points.*

The proof of this interesting theorem is credited to Ahlfors (1964). More modern proofs of the theorem exist, like the one due to Marden (2006). As can be inferred from the rationale above, we do not attempt the proof of this theorem in this document, which is fundamentally oriented to the Limit Set. [1]

[1]At the moment of editing this work, the author realizes that this theorem has a difficult position in most

The Ahlfors Finiteness theorem allows addressing the following noteworthy results.

Theorem 35. *Suppose G is finitely generated and not elementary, and $\Omega(G) \neq \emptyset$. Then, each component of $\Omega(G)$ is either simply or infinitely connected.*

Proof. To make the proof let us assume that $\Omega(G)$ is finitely–but not simply–connected, and look for a contradiction.

In this proof we utilize results appearing in Marden (2006) and Marden (2007). Those results are a consequence of the Ahlfors' Finiteness theorem (Th. 34). The utilized result is that $\Omega(G)$ is preserved by an element g of infinite order.

Once origin and pertinence of this result is clarified, we may proceed immediately to carry out the proof.

Choose a loop σ that separates the boundary components. The sequence of loops $\{g^k(\sigma)\}$ converges to the fixed points of g. On one side, the loops $g^k(\sigma)$ separate the boundary components of $\Omega(G)$. On the other, they converge to fixed points. Thus, the fixed points are limits of infinitely many boundary components of $\Omega(G)$. We have reached a contradiction, which proves the statement in question. □

Theorem 36. *Suppose G is finitely generated and not elementary, and $\Omega(G) \neq \emptyset$. $\Omega(g)$ has one, two or infinitely many components.*

Proof. $\Omega(G) \neq \emptyset$ implies that there is at least one component in the discontinuity set.

Suppose there are finitely many components, $\Omega_1, \Omega_2, \ldots, \Omega_m$. We arbitrarily set that the component in which ∞ is located is the Ω_m. By the same arguments utilized in the proof of the Theorem 35, there is a subgroup G_0 of finite index that preserves each of them.

Let now take a loxodromic transformation $g \in G_0$. g preserves Ω_1 and Ω_2. If p and q are the fixed points of g we may find arcs $\sigma_1 \in \Omega_1$ and $\sigma_2 \in \Omega_1$ such that $\left(\cup_{k=\infty}^{\infty} g^k(\sigma_1)\right) \cup \left(\cup_{k=1}^{\infty} g^k(\sigma_2)\right) \cup \{p,q\}$ is a closed loop that passes through p and q, and that only traverses the limit set at this point. This curve divides Ω into two sets, U and U', where ∞ is located in U. Let us consider another loxodromic element, h, of G_0 that will have an attracting point inside of U. The arc, τ, between ∞ and $h(\infty)$ must be in Ω_m because of the conservation of the components. But the arc $\cup_{k=1}^{\infty} h^k(\infty)$ connects ∞ to the fixed point of $h \in \Omega_1 \cup \Omega_2$: a contradiction. □

Example 8. *We may check the previous theorem observing Figures 4.4, 5.7 and 5.6. The limit set of Figure 4.4 has one component, and resembles Cantor's Set. Figure 5.7 shows a Quasi-Fuchsian limit set with two components. The Apolonian Gasket of Figure 5.6 divides Ω into infinitely many components.*

common references. Beardon (1983), maybe because of the orientation of his book, does not mention it. Marden (2007) formulates it but does not give a proof. Maskit (1988) mention it but does not outline it. Matsuzaki and Taniguchi 1998 certainly provide good frames for this theorem, but devote a significant part of their book, sections 4.1 and 4.2, to the introduction and the corresponding proof of the theorem.

We conclude this small digression outlining two theorems that appear in Marden (2007), and that we do not endeavor to prove.

Theorem 37. *If each of two components* $\Omega_1, \Omega_2 \in \Omega(G)$ *is preserved by* G *then each one is simply connected and* $\Omega(G) = \Omega_1 \cup \Omega_2$.

Theorem 38. *If one component of* Ω *of* $\Omega(G)$ *is preserved by* G*, all the others are simply connected.*

Those two last results are of significant importance and interest for the study of Quasi-Fuchsian groups.

We may also try to interpret some of our examples with them.

Figure 5.7 shows a limit set forming a Jordan curve dividing $\Omega(G)$ into two components Ω_1, Ω_2. Theorem 37 implies that if a group G is such that $G(\Omega_1) = \Omega_1$ and $G(\Omega_2) = \Omega_2$, those are the unique simply connected components.

The application of theorem 38 to the Apollonian Gasket of Figure 5.6 is of greater interest. In the case that one of the circles, free of limit points, could be conserved (hypothetically), all the other infinite circles that form the reticle of the Gasket would be also simply connected.

APPENDIX C

Python Code Generated for the Calculation of Schottky group Transformations. Circle's Method

This annex contains and describes the numerical code utilized for the generation of Example 2.

Explanation of the Structure of the Code

The code is written in the computer language Python (Rossum, 1995) utilizing the numerical module Numpy (Walt et al., 2011) for the computationally demanding parts. The results obtained are plotted utilizing Matplotlib library (Hunter, 2007), that allows for polygraphic quality output while keeping high simplicity for the programming.

The code is based on the fact that the Möbius transformation of a circle is indeed another circle.

The main structure of the code is as follows:

1. The circles utilized to calculate the generators of the group are given in the form of coordinates of the centers and radius.

2. The code calculates the generators of each transformation based on the circle's pairing algorithm as explained in example 2.

3. These circles and the generators are provided to the function that will create the transformations.

4. The transformations will be applied utilizing the following infrastructure:

   a) A routine that calculates the transformation: *mobius_on_circle*.

   b) The *circulo* class, which contains the description of the circle to which the transformation with be applied.

   c) A collection of circles in the form of the *circulos* class. The *circulos* class also contains the methods to manipulate the circles. Namely, to make a transformation and to store the transformations in a collection.

5. The transformations and collection of circles are carried out and stored in terms of level. By *levels* we understand the amount of letters that the words of the transformations contain.

6. The results are plotted utilizing *matplotlib* library.

The different sections of the code contain comments allowing an easy inspection of the different components program. The different sections are mostly self-explanatory.

Python code

```
#!/usr/bin/python

import numpy as np
import matplotlib.pyplot as plt
import matplotlib.cm as cm

class circulo:

### This class contains the center and radius of the circle and performs
### a Moebius trasnformation on it
    def __init__(self, centro, radio):
        self.centro=centro
        self.radio=radio

    def rad(self):
        return self.radio

    def cen(self):
        return self.centro

#### Moebius transformation following Indra's Pearls, Mumford et al. (see refs)
    def mobius_on_circle(self, matrix):
        self.aux=matrix[1,1]/matrix[1,0]+self.centro
```

```
        self.z=self.centro-self.radio*self.radio/self.aux.conj()
        self.centro_new=(matrix[0,0]*self.z+matrix[0,1])/(matrix[1,0]*self.z+matrix[1,1])
        self.rr=self.radio+0j
        self.aux2=self.centro_new-(matrix[0,0]*(self.centro+self.rr)+matrix[0,1])/(matrix[1,0]*(self.centro+self.rr)+matrix[1,1])
        self.radio_new=np.abs(self.aux2)
        return (self.centro_new, self.radio_new)

class circulos:
#### This clas is a collection of circles
#### Also when the circles and the transformations
#### are given, it calculates the levels requested
#### and collected the trasnformed circles

#### Constructor initialize the class

    def __init__(self, cir, trans_i, tags_i):
        self.cir_ini=cir
        self.trans_ini=trans_i
        self.tags_ini=tags_i
        self.trans=[]
        self.tags=[]
        self.circs=[]
        self.tags.append(self.tags_ini)
        self.trans.append(self.trans_ini)
        self.circs.append(self.cir_ini)

#### Calculates one level of transformations
    def calc_trans(self, level):
        self.trans_lev=[]
        self.tags_lev=[]
        for j in range(len(self.trans_ini)):
            for i in range(len(self.trans[level-1])):
                if self.tags[level-1][i]!=j:
                    self.trans_lev.append(self.trans[level-1][i]*self.trans_ini[j])
                    self.tags_lev.append(self.tags_ini[j])
```

```
        self.trans.append(self.trans_lev)
        self.tags.append(self.tags_lev)

#### Calculates all levels of transformations
    def calc_levels_trans(self, lev):
        for i in range(lev):
            self.calc_trans(i+1)

#### Calculates the circles and append them
    def calc_circles_level(self, lev):
        self.cir_loc=[]
        for j in range(len(self.cir_ini)):
            for i in range(len(self.trans[lev])):
                if self.tags[lev][i]!=j:
                    (self.cen_t, self.r_t)=self.cir_ini[j].
    mobius_on_circle(self.trans[lev][i])
                    self.cir_loc.append(circulo(self.cen_t, self.r_t))

        self.circs.append(self.cir_loc)

#### Calculates all levels of circles
    def calc_levels_cirs(self, lev):
        for i in range(lev):
            self.calc_circles_level(i+1)

#### Number of levels to be calculated
levels=8

#### Auxiliary variables for centers and radius
k=1
x=np.sqrt(2)
u=x
v=1
y=v

### Initial circles given
Ca=circulo( 0+k*u/v*1j, k/v)
CA=circulo( 0-k*u/v*1j, k/v)
```

```
Cb=circulo( x/y+0j , 1/y)
CB=circulo(-x/y-0j , 1/y)

### One transformation and its inverse
a1=np.matrix([[1+0j,-Ca.cen()],[0+0j,1+0j]])
a2=np.matrix([[0+0j,Ca.rad()*CA.rad()+0j],[1+0j,0+0j]])
a3=np.matrix([[1+0j,CA.cen()],[0+0j,1+0j]])

aa=a3*a2*a1
AA=aa**(-1)

### Second transformation
b1=np.matrix([[1+0j,-Cb.cen()],[0+0j,1+0j]])
b2=np.matrix([[0+0j,Cb.rad()*CB.rad()+0j],[1+0j,0+0j]])
b21=np.matrix([[0+0j,Cb.rad()+0j],[1+0j,0+0j]])
b22=np.matrix([[-1+0j,0+0j],[0+0j,1+0j]])
b23=np.matrix([[CB.rad()+0j,0+0j],[0+0j,1+0j]])
b3=np.matrix([[1+0j,CB.cen()],[0+0j,1+0j]])

bb=b3*b2*b1
bb=b3*b23*b22*b21*b1
BB=bb**(-1)

#### initialize the class
#### Circles, transformations, tags
#### tags are an index that each circle keeps to ovoid to
#### apply to itself its own transformation
#### aa transform Ca into CA...
circs=circulos([Ca, Cb, CA, CB], [aa,bb,AA,BB], [2,3,0,1])

#### Calculation of transformations and circles
circs.calc_levels_trans(levels)
circs.calc_levels_cirs(levels)


#### Plotting the output
colores=['b','g','r','c', 'm', 'y', 'firebrick', 'gold', 'pink', 'orange'
    , 'tan']
```

```
circle1=[]
for l in range(len(circs.circs)):
    for n in range(len(circs.circs[l])):
        circle1.append( plt.Circle((circs.circs[l][n].cen().real, circs.circs[l][n].cen().imag), circs.circs[l][n].rad(), color=colores[l]))

fig = plt.gcf()

ax = plt.gca()
plt.axis('off')

ax.set_xlim((-3,3))
ax.set_ylim((-3,3))
for n in range(len(circle1)):
    fig.gca().add_artist(circle1[n])

fig.savefig('plotcircles.pdf', format='pdf')
```

APPENDIX D

Python/Cython Code Developed to Calculate Transformations of a Set of Points

This annex contains the Python (Rossum, 1995) and Cython (Behnel et al., 2011) codes generated for the creation of Example 3.

The structure of a mixed Cython/Python code is simple. The Cython module is compiled utilizing a *setup* file in order to generate a *shared object*. The shared object is then called by the pure Python module. The code generated gains very significantly in efficiency, allowing for higher performance and increased control of the memory. At the same time, the productivity of the code development, typical of Python language, is almost kept. A much deeper insight into the procedure of writing a Cython code can be gained by consulting Behnel (2017).

We may now proceed to explain the structure of our particular development.

1. The Python routine has the following structure:

    a) The base circles are given in form of coordinates of the centers and radii. This way of proceeding is inherited from our previous program. It is kept exclusively in order to ensure that we are dealing with Kleinian groups, fulfilling all necessary properties.

       The code calculates the generators of the transformation based on the circle's pairing algorithm as explained in Example 3.

    b) A set of four transformations of the origin, the single base-point we utilize, is obtained and provided to the Cython module. Those constitute the initial seed of the program. The cancellation of a transformation and its inverse, that may provide spurious limit points, is avoided by keeping track of the inverse of the last transformation applied.

       The Cython module will calculate all the transforms of the set of points given.

    c) Once the whole set of the images of the base-point are available, they are plotted.

2. The Cython routine is a collection of independent functions. Those are:

   a) A function, *transform*, to carry out the Möbius transformation of a single point.

   b) A function, *apply_trans_comp*, that performs the Möbius transformation of the whole set of given points utilizing the function *transform*.

   c) A function, *obtain_points*, that orchestrates the successive application of the *apply_trans_comp* function in order to obtain all possible transforms considering words of a certain number of letters.

D.1 Python routine

```

#!/usr/bin/python

import numpy as np
import matplotlib.pyplot as plt
import matplotlib.cm as cm
import generador_fast
import time

# center and radius
# of initial circles
# for the generation of transformation

#c1: i, srqt(3)/2
#c1': -i, srqt(3)/2
#c2: 1/2, sqrt(2)/2
#c2': -1/2, sqrt(2)/2

# routine for the Moebius transformation
# given point ans transform output result

def transform(point, matrix):
return (matrix[0,0]*point+matrix[0,1])/(matrix[1,0]*point+matrix[1,1])

# track time of execution
start_time = time.time()

```

```python
# calculate generators of transformation

a1=np.matrix([[1+0j,0-1j],[0+0j,1+0j]])
a2=np.matrix([[0+0j,np.sqrt(3)/2*np.sqrt(3)/2+0j],[1+0j,0+0j]])
a21=np.matrix([[0+0j,np.sqrt(3)/2+0j],[1+0j,0+0j]])
a22=np.matrix([[-1+0j,0+0j],[0+0j,1+0j]])
a23=np.matrix([[np.sqrt(3)/2+0j,0+0j],[0+0j,1+0j]])
a3=np.matrix([[1+0j,0-1j],[0+0j,1+0j]])

b1=np.matrix([[1+0j,-1/2+0j],[0+0j,1+0j]])
b2=np.matrix([[0+0j,np.sqrt(2)/2*np.sqrt(2)/2+0j],[1+0j,0+0j]])
b21=np.matrix([[0+0j,np.sqrt(2)/2+0j],[1+0j,0+0j]])
b22=np.matrix([[-1+0j,0+0j],[0+0j,1+0j]])
b23=np.matrix([[np.sqrt(2)/2+0j,0+0j],[0+0j,1+0j]])
b3=np.matrix([[1+0j,-1/2+0j],[0+0j,1+0j]])

# generators are

a=a3*a2*a1
#a=a3*a23*a22*a21*a1
b=b3*b2*b1
#b=b3*b23*b22*b21*b1


# Calculation of transformations inverses

A=a**(-1)
B=b**(-1)

# array of transformations (generators)

trans=np.complex128(np.array([a,A,b,B]))

# array of logs of generators
# to avoid applying an inverse to a
# transformation (canceling)
# wrong inverse

```

```
logs=np.int32(np.array([1,0,3,2]))

# number of levels (size of words)

levels=np.int32(10)

P=np.complex(0,0)

# initial transforms

p0=transform(P, a)
p1=transform(P, A)
p2=transform(P, b)
p3=transform(P, B)

# initial set of points and logs
points=np.complex128(np.array([p0,p1,p2,p3]))
points_log=np.int32(np.array([1,0,3,2]))

# obtain points solutions of the transformations

(points, points_log) = generador_fast.obtain_points(points, points_log,
    trans, logs, levels)

# calculation is finished
# print starts
print "CALCULATION ENDED"
fig = plt.gcf()

# generate canvas for the ploting
ax = plt.gca()
plt.axis('off')

#ax.set_xlim((-1.00,1.00))
#ax.set_ylim((-1.00,1.00))

# non deformation inc canvas
plt.gca().set_aspect('equal', adjustable='box')
```

```

# separate real and imaginary parts
X = [x.real for x in points]
Y = [x.imag for x in points]

# plot and save
plt.scatter(X,Y, s=0.01, color='black')
fig.savefig('plotpoints.pdf', format='pdf')

# execution time
print("--- %s seconds ---" % (time.time() - start_time))
```

D.2 Cython module

```
#!/usr/bin/python

import numpy as np
import time

cimport numpy as np

# cython module
# containing
# several functions

# routine for the Moebius transformation
# given point ans transform output result

cdef np.complex128_t transform(np.complex128_t point, np.ndarray[np.
    complex128_t, ndim=2] matrix):
return (matrix[0,0]*point+matrix[0,1])/(matrix[1,0]*point+matrix[1,1])

# routine that apply transformations to the points given
# track is kept in "log" array of inverse of last transformation
# to avoid appliying canceling transformations

# given a set of points and logs output the transfomred points
# and their logs

```

```
cdef apply_trans_comp(np.ndarray[np.complex128_t, ndim=1] points, np.ndarray[np.int32_t, ndim=1] points_log, np.ndarray[np.complex128_t, ndim=3] trans, np.ndarray[np.int32_t, ndim=1] logs):

local_points=(1+0j)*np.zeros(points.size*(logs.size-1), dtype=np.complex128)
local_log=np.zeros(points.size*(logs.size-1), dtype=np.int32)

cdef int count = 0

for i in range(points.size):
for j in range(logs.size):
if points_log[i] != j:
local_points[count]=transform(points[i], trans[j])
local_log[count]=logs[j]
count=count+1

return (local_points, local_log)


# given a set of initial points and logs (initial seed)
# given intial transfomations and their logs
# and given number of levels
# outputs the set of points after
# number of levels iterations

def obtain_points(np.ndarray[np.complex128_t, ndim=1] points, np.ndarray[np.int32_t, ndim=1] points_log, np.ndarray[np.complex128_t, ndim=3] trans, np.ndarray[np.int32_t, ndim=1] logs, np.int32_t levels):

for i in range(levels):
local_points=(1+0j)*np.zeros(0, dtype=np.complex128)
local_log=np.zeros(0, dtype=np.int32)

(local_points, local_log)=apply_trans_comp(points, points_log, trans, logs)

points=np.array(local_points, copy=True)
```

```
points_log=np.array(local_log, copy=True)

return (points, points_log)
```

D.3 Setup file

```
from distutils.core import setup
from Cython.Build import cythonize

setup(
ext_modules = cythonize("generador_fast.pyx")
)
```

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